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On the Dimension of Spline Spaces, a Homological Approach

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Introduction

Abstract

One of the puzzlingly hard problems in *Computer Aided Geometric Design* and *Approximation Theory* is that of finding the dimension of the spline space of C^r piecewise polynomials of degree n over a planar complex Δ . We denote such spaces by $S_n^r(\Delta)$. In this thesis, we use homological methods, developing a new tool, to compute the dimension of $S_n^r(\Delta)$.

Historical Remark

In 1965 Olav Arnfinn Laudal published the article, *Sur les limites projectives et inductives*, in *Annales scientifiques de l'É.N.S.* [1], where he gave a general theory of limits of projective systems on partially ordered sets. His treatise however considered mainly constant projective systems. He didn't expect this to be applied in other part of mathematics, and in the 1980's he announced to his student at the time, Arne B. Sletsjøe, how the work done to mathematics in general in the 1960's mainly was "for abstract purposes only". However, it has proven itself to be usefull in other areas.

In 2013 Sletsjøe began considering Laudal's work and wether or not this could be applied. It resulted in the manuscript *Cohomology of projective systems on ranked posets* [2](2015), where he considered more general projective systems, different cohomology theories for ranked posets with local orientation and showed how the cohomology theories under certain conditions concided.

In 1946, almost twenty years before Laudal published his article, Schoenberg did the first mathematical reference to splines in *Contributions to the Problem of Approximation of Equidistant Data by Analytic Functions* [3].

As time went, the subject of *Computer Aided Geomtric Design* grew rapidly and the study of splines turned out to become quite usefull. An example of this is the study of methods and algorithms for the descriptions

of shapes; called *Geometric Modeling*. It usually involves piecewise algebraic representations of shapes, where the effective treatment of these representations of shapes leads to the resolution of polynomial systems of equations, which requires the use of stable and efficient tools.

Within the numerical analysis community, the use of higher order polynomial representations has been conceived as a new way to break the complexity barrier caused by piecewise linear representations, and to deal efficiently with free-form geometry.

The motivation for the use of splines in the algebraic representations of surfaces and modeling of curves was; the fact that using just one polynomial does not give much flexibility. Instead, one should use piecewise polynomial functions, which possesses a certain degree of smoothness in the subspaces where the polynomial pieces connect, to approximate larger regions of a model. This keeps the polynomial degree lower and allows more flexible approximations.

As usual in mathematics, also in the theory of splines some difficulties arose. One of them is the well know problem of determining the space of piecewise polynomial functions with a given degree of smoothness. In particular, the dimension of the space has been discussed by several authors, by various methods. See e.g. [4], [5], [6], [7], [8], [9] and [10].

Introduction

The goal of this thesis is developing a new theory for computing the dimension of the space of piecewise polynomial functions. The work of Sletsjøe and Laudal will be the framework for the thesis and we will see how their work is going to be useful in computing dimensions of spline spaces, especially how we can use (co-)homology theory on posets of a planar complex Δ to compute its dimension. That would say; the general framework for our set-up is a partially ordered set Λ and a projective system F of abelian groups on Λ . The partially ordered set gives the combinatorial structure of a grid or a triangulation of an underlying space. The projective system is a system of polynomial functions of bounded degree, where smoothness is encoded algebraically in the restriction of the system to the connection subspaces.

Short description of each chapter:**Chapter 1**

Contains the basic definition of splines and spline spaces, this section is basically meant for those readers who are not familiar with the terminology of splines.

Chapter 2

We give the formal definition for our framework, including e.g. partially ordered sets, orientation and (co-)homology theory. At the end of the chapter we establish the formal relation between spline spaces and (co-)homology theory.

Chapter 3

To compute the dimension of complex/large spline spaces, we need to know how to compute the dimension of its subspaces, therefore we give proofs of the dimension of some of these subspaces.

Chapter 4

We introduce one last tool to compute the dimension of a larger spline space than those in chapter 3. Further we consider the dimension of the Morgan-Scott triangulations. Which is one of the prime examples for computing the dimension of spline spaces. Lastly, we reveal how to compute this dimension, in some of its cases.

Chapter 1

Splines

For the readers without prior knowledge of splines it may be useful to have a formal definition and an explanation of splines. This and a basic understanding of their existence is given in this chapter. However this chapter may be skipped by those who already feel familiar with splines.

What are splines? And what do they look like?

Let us consider the simplest form of a spline; consider the interval $[a, c] \subset \mathbb{R}$ and a subdivision $[a, b] \cup [b, c]$, $a < b < c$. Then given two polynomials $f_1, f_2 \in \mathbb{R}[x]$, a spline function $f(x) \in \mathbb{R}[x]$ takes the form:

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in [a, b] \\ f_2(x) & \text{if } x \in [b, c] \end{cases} \quad (1.1)$$

Of course we can make a trivial spline function just by choosing the same polynomial $f_1 = f_2$. This, however, wouldn't contribute to any new theory. From elementary calculus we see that the spline function in (1.1) is a continuous function on $[a, c]$ if and only if $f_1(b) = f_2(b)$.

Let us denote by a **C^r function** f , a function f which is r -times differentiable and its r -th derivative, $f^{(r)}$, is piecewise continuous. All polynomials $f \in \mathbb{R}[x]$ are by construction C^∞ .

Since f consists of the two polynomials f_1 and f_2 , and their derivatives are also polynomials we can consider piecewise polynomial derivative functions

$$f^{(r)}(x) = \begin{cases} f_1^{(r)}(x) & \text{if } x \in [a, b] \\ f_2^{(r)}(x) & \text{if } x \in [b, c] \end{cases}$$

for $r \geq 0$. As above, f is a C^r function on $[a, c]$ iff. $f_1^{(s)}(b) = f_2^{(s)}(b)$ for each s , with $r > s \geq 0$.

We could of course have chosen $f_1(x)$ and $f_2(x)$ to be spline functions, by induction we would then have defined all spline functions in the plane.

The formal definition follows:

Definition 1.0.1. A **spline function** is a piecewise polynomial function, which possesses a sufficiently high degree of smoothness at the places where the polynomial pieces connect.

The notion of the spline function (1.1) can easily be generalized to higher dimension by the ordered sequence $(f_1, f_2, \dots, f_n) \in \mathbb{R}[x]^n$, and the C^r splines form a vector subspace of $\mathbb{R}[x]^n$, under the usual componentwise addition and scalar multiplication. It is, however, common to consider spline functions where the degree of each component is bounded by some fixed integer k . That space is denoted by C_k^r and it is also a vector subspace of $\mathbb{R}[x]^n$.

Chapter 2

The Set-Up

In this chapter we give a base for the needed knowledge and results for this thesis. Most of the results are taken from Sletsjøe's manuscript *Cohomology of projective Systems on ranked Posets* [2] and Laudal's article *Annales scientifiques de l'É.N.S.* [1].

The proofs and methods of this chapter build upon algebraic geometry, and we assume some familiarity to commutative algebra and (co-)homology theory.

2.1 Posets, orientation and the order complex

We give a little motivation for the quite machinery coming. Let us first of all remember the goal of this thesis, namely finding the dimension of spline spaces over geometric complexes.

What is a geometric complex and what does it look like?

Most of the time we will have complexes consisting of triangulations of the plane \mathbb{R}^2 , so let us consider a triangulated plane area in this example. We may however use any combination of polygons, they are all topological equal, even though they differ some in structural equality. The figure below illustrates how such a complex could look like.

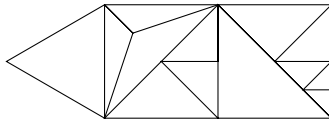


Figure 2.1: An example of an triangulated mesh.

What is this complex built up by?

Cells. A **cell** is the generalization of the notion of a polygon or polyhedron to arbitrary dimensions. Specifically, a k -cell is a k -dimensional polytope which is the convex hull of its $k + 1$ vertices. For example a 0-cell is a vertex/single point, a 1-cell is an edge/line and a 2-cell is a triangle/planar object. We may glue the cells together as in figure 2.1. If we now construct a set, by taking all cells in figure 2.1 to be in the set, we have a cell complex.

Definition 2.1.1. A **cell complex** \mathcal{K} is a set of cells that satisfies the following conditions:

1. A face of a cell from \mathcal{K} is also in \mathcal{K} .
2. The intersection of two cells $\sigma_1, \sigma_2 \in \mathcal{K}$ is a face of both σ_1 and σ_2 ,

where a **face of a cell** is said to consist of those elements used in constructing the cells, e.g. in the case of a two dimensional planar cell, its faces are lines and vertices.

We would like to adopt these geometric notions into algebraic notions. Hence we need our next definition. Our goal during the current section is being able to take the necessary information out of a cell complex and express it algebraically. A structure helping us with this is the notion of a partially ordered set:

Definition 2.1.2. A **partial order** is a binary relation \leq over a set Λ which is **antisymmetric**, **reflexive** and **transitive**. A set with a partial order is called a **partially ordered set** or a **poset**. A poset Λ is **connected** if for any pair of elements $\lambda_a, \lambda_b \in \Lambda$, there exists a finite sequence $\lambda_a = \lambda_0, \lambda_1, \dots, \lambda_n = \lambda_b$ such that

$$\lambda_i < \lambda_{i+1} \text{ or } \lambda_{i+1} < \lambda_i \quad \forall \quad i = 0, \dots, n-1$$

Example 2.1.3. Two examples of posets are:

1. \mathbb{R} is a connected poset ordered by the standard "less than or equal" relation \leq .
2. The set of subspaces of a vector space ordered by inclusion, \subseteq . ★

We would like to, in a structured way, have some notion of which of the elements in a posets are *equal in size*, we therefore need to define the notion of dimension and the q -skeleton of a poset Λ .

Definition 2.1.4. A **ranked poset**, is a finite, connected poset, equipped with a strict order-preserving dimension function $d : \Lambda \rightarrow \mathbb{N}_0$.¹ The minimum value is called the **minimal rank** of Λ , denoted m_Λ , and the maximum

¹**Notation remark:** $\mathbb{N}_0 = \{\mathbb{N} \cup 0\}$

value, d_Λ , of the dimension function is called the **geometric dimension** of the poset. We say that Λ is of **pure minimal rank** if all minimal elements have the same dimension, and of **pure dimension** if all maximal elements of Λ have the same dimension. Elements $\lambda \in \Lambda$ of dimension $q = d(\lambda)$ are called **q -cells** of Λ . And the set $\Lambda_q \subset \Lambda$ of all q -cells is the **q -skeleton** of Λ .

We illustrate by an example how splines and posets are linked together. Representing the poset by a Hasse diagram:

Example 2.1.5. Let Δ be one of the inner T -meshes in \mathbb{R}^2 given in [5], e.q. the figure below. We construct the poset of Δ . For simplicity we redraw the original T-mesh:

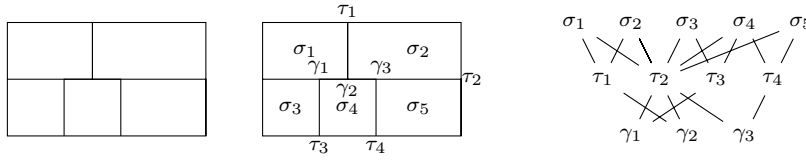


Figure 2.2: A T-mesh given names one each of its cells, and its poset represented by a Hasse diagram to the right.

By denoting all the 2-cells σ_i , all 1-cells τ_i and all 0-cells γ_i in Δ . We see that we get the poset drawn to the right in figure 2.2, where for $\lambda_i, \lambda_j \in \Lambda$ we have $\lambda_i \leq \lambda_j$ iff. either λ_i is a square and λ_j is one of its vertices or edges, or λ_i is an edge and λ_j is one of its vertices.

In fact this is not quite correct because τ_2 actually consists of the 4 edges: defined from boarder to γ_1 , γ_1 to γ_2 , γ_2 to γ_3 and γ_3 to boarder. However they are shown as one to simplify the illustration². ★

Definition 2.1.6. If $\lambda_1 < \lambda_2$ we say that λ_1 is a **facet** of λ_2 of codimension $d(\lambda_2) - d(\lambda_1)$. A facet $\lambda_1 < \lambda_2$ of codimension 1 is a **face** of λ_2 . For any cell λ in Λ , $\Delta\lambda$ denotes the set of faces of λ and for any subset $S \subset \Lambda$, $\Delta S = \{\tau \in \Delta\sigma \mid \text{for some } \sigma \in S\}$. The set of facets of codimension k is denoted $\Delta^k\lambda$, while the set of all facets is denoted by $\Delta^\bullet\lambda$. In the same manner, the cells that have λ as one of its faces are called a **co-face** of λ . The set of cofaces of λ is denoted $\nabla\lambda$.

Definition 2.1.7. The **dual poset** Λ^* of a poset Λ has the same objects as Λ , but with reversed ordering, i.e. $\lambda_1 < \lambda_2$ holds in Λ^* iff. $\lambda_2 < \lambda_1$ holds in Λ .

We use the notation $\lambda^* \in \Lambda^*$ for the corresponding object of $\lambda \in \Lambda$ in Λ^* . The dual S^* of a subset $S \subset \Lambda$ is the set $S^* = \{\lambda^* \mid \lambda \in S\}$ of dual elements.

²They will later on, in most cases, coincide as we assign the same projective system to all the 4-parts of τ_2 .

If Λ is ranked, there is also a natural ranking of Λ^* ; the dimension of the dual element λ^* is the codimension of λ , i.e. $d(\lambda^*) = d_\Lambda - d(\lambda)$, where d_Λ is the geometric dimension of Λ . Thus the dual complex Λ^* has geometrical dimension $d_{\Lambda^*} = d_\Lambda - m_\Lambda$ and minimal rank $m_{\Lambda^*} = 0$.

Figure 2.3 illustrate an example of a dual poset:

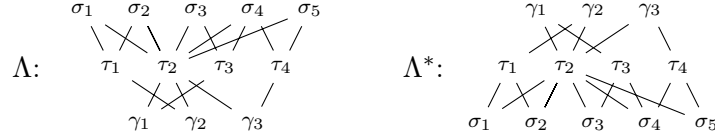


Figure 2.3: A poset Λ and its dual poset Λ^* .

The q -skeleton of Λ corresponds to the $(d_\Lambda - q)$ -skeleton of Λ^* , so it follows that:

Proposition 2.1.8. Given $\lambda \in \Lambda_p$, then $\nabla \lambda^* = (\Delta \lambda)^*$:

Proof.

$$(\Delta \lambda)^* = \{\sigma \in \Lambda_{p-1} \mid \lambda > \sigma\}^* = \{\sigma^* \in (\Lambda^*)_{d_\Lambda - (p-1)} \mid \sigma^* > \lambda^*\} = \nabla \lambda^*$$

□

Corollary 2.1.9. We have $\Lambda^{**} \simeq \Lambda$. Where the isomorphism $\phi : \Lambda \rightarrow \Lambda^{**}$ is given by a shift in the dimension m_Λ . We have equality $\Lambda^{**} = \Lambda$ if and only if Λ has minimal rank $m_\Lambda = 0$.

To do our calculations on posets, we have to restrict our posets somehow, thereby we introduce the notion of an abstract cell complex, we define it the way it was defined in [2]:

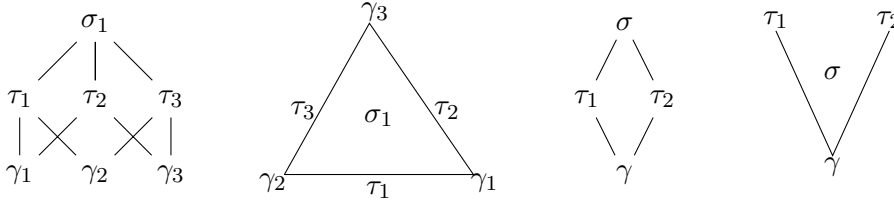
Definition 2.1.10. An **abstract cell complex** is a ranked poset satisfying the following condition: For any $\lambda_1 \in \Delta^2 \lambda_2$, there exists exactly two elements in $\Delta \lambda_2$, τ_1 and τ_2 , such that $\lambda_1 < \tau_i < \lambda_2$, $i = 1, 2$, i.e. $\Delta \lambda_2 \cap \nabla \lambda_1 = \{\tau_1, \tau_2\}$.

For those readers familiar with the notion of cellular (co-)homology, they will quite certain see the necessity of an orientation to an abstract cell complex. It is quite obvious that a cell complex with a geometric realisation, know as a **geomtric cell complex**, do have a natural oritentation, but these are not necessarily abstract cell complexes.

But what is a geometric realisation? And which cell complexes have them?

We say that a cell complex is **geomtric realizable** if the closure of the cell complex equals the cell complex. I.e. given a cell complex Λ , Λ is said to have a geomtric realisation if $\overline{\Lambda} = \Lambda$.

Let's illustrate this. The cell complex to the left below have a geomtric realisation, given in the middle left in the illustration. But the middle right figure do not have an geomtric realisation. It is however an open cell complex of an geometric realizable cell complex, shown to the right.



Remark 2.1.11. In all cases we are considering it looks like the abstract cell complex restriction to a cell complex, is the restriction needed to say that given an abstract cell complex Λ_1 , then Λ_1 is a subset of a geometric cell complex.

I.e. there exist a geometric cell complex Λ such that $\Lambda_1 \subseteq \Lambda$ and $\overline{\Lambda_1} = \Lambda$. Hence to any given abstract cell complex, we could just have added some points, for it to be a geomtric cell complex. And instead of imposing the orientation below we could have imposed a natural orientation by letting it be the orientation of its "geometric closure".

Unfortunately the idea came up to late in the prosses with the thesis and there have not been time to prove it, but it seems quite reasonable. ★

Since there exists abstract cell complexes not having a geometric realisation, we can not be certain that an abstract cell complex necessarily has a natural orientation. However, in order to define cellular (co-)homology of an abstract cell complex, we need some sort of orientation. We therefore impose a local orientation, valid for all abstract cell complex Λ . To do so, we first need the notion of a cover in Λ .

Definition 2.1.12. If λ_1 is a face of λ_2 , then the pair $\lambda_1 < \lambda_2$ is called a **cover** in Λ , and the set of all covers in Λ is denoted $\mathbf{Cov}(\Lambda)$.

We are now ready to define our orientation:

Definition 2.1.13. A **local orientation** of an abstract cell complex Λ is a map $\epsilon : \mathbf{Cov}(\Lambda) \rightarrow \{\pm 1\}$ such that for $\lambda_1 \in \Delta^2 \lambda_2$, with $\Delta \lambda_2 \cap \nabla \lambda_1 = \{\tau_1, \tau_2\}$ we have

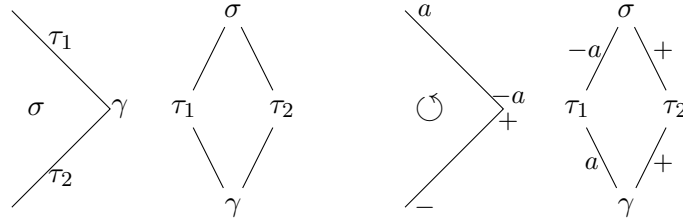
$$\epsilon_{\lambda_1 < \tau_1} \epsilon_{\tau_1 < \lambda_2} + \epsilon_{\lambda_1 < \tau_2} \epsilon_{\tau_2 < \lambda_2} = 0$$

where $\epsilon_{\lambda < \tau} = \epsilon(\lambda < \tau)$.

What does a natural orientation look like?

We illustrate it with an example:

Example 2.1.14. In the two dimensional case, the only time we have $\gamma \in \Delta^2\sigma$, with $\Delta\sigma \cap \nabla\gamma = \{\tau_1, \tau_2\}$ is when we have a 2-cell over a vertex. Hence it looks like the leftmost figure:



Its poset is shown to the middle right in the figure. We first start by deciding then decide an orientation of the two-cell σ , either \circlearrowleft or \circlearrowright , say \circlearrowleft . Secondly, to satisfy 2.1.13, we need to choose either positive or negative orientation on the edges. Meaning, an edge could either go to $+$ or $-$ in the orientation, e.g. say τ_2 from $-$ to $+$ see figure above.

We now have $\epsilon_{\gamma < \tau_2} \epsilon_{\tau_2 < \sigma} = 1 * 1 = 1$, so to satisfy the local orientation property we have to have $\epsilon_{\gamma < \tau_1} \epsilon_{\tau_1 < \sigma} = -1$. Hence either $\epsilon_{\gamma < \tau_1} = -1$ and $\epsilon_{\tau_1 < \sigma} = 1$ or conversely. This is satisfied by choosing a , in the figure above, either equal to $+1$ or -1 . ★

We could also leave some extra restrictions on an abstract cell complex to make it more likely for it to have a geometric realisation:

Definition 2.1.15. An abstract cell complex Λ of pure dimension n is called **non-singular** if each 1-cell of Λ has at most two vertices and each $(n-1)$ -cell of Λ is a face of at most two maximal cells.

Definition 2.1.16. A local orientation of a non-singular abstract cell complex is an **orientation** of Λ if for any $(n-1)$ -cell τ with $\nabla\tau = \{\sigma_1, \sigma_2\}$ we have $\epsilon_{\tau < \sigma_1} = -\epsilon_{\tau < \sigma_2}$, and for each 1-cell τ with $\Delta\tau = \{\gamma_1, \gamma_2\}$ we have $\epsilon_{\gamma_1 < \tau} = -\epsilon_{\gamma_2 < \tau}$. The abstract cell complex is **orientable** if there exist such an orientation.

Since not all of our cohomology theory will be based upon points in the posets but its chains of elements, we should define a set from Λ , where the elements are chains of elements in Λ .

Definition 2.1.17. The **order dimension** $r(\Lambda)$ of a finite poset Λ is the maximum length of ordered chains of elements, i.e.

$$r(\Lambda) = \max\{p \mid \lambda_0 < \lambda_1 < \dots < \lambda_p \in \Lambda\}$$

Since the dimension function of Λ is strictly order-preserving, it follows that $d_\Lambda \geq r(\Lambda)$. For a poset Λ we denote by $\Lambda^{(1)}$ the induced **order complex** of Λ .

Further the p -skeleton of $\Lambda^{(1)}$ consists of sequences

$$\lambda_0 < \lambda_1 < \cdots < \lambda_p$$

While the order complex $\Lambda^{(1)}$ is equipped with the local orientation ϵ given by

$$\epsilon(\lambda_0 < \cdots \hat{\lambda}_i \cdots < \lambda_p \prec \lambda_0 < \cdots < \lambda_i < \cdots < \lambda_p) = (-1)^i$$

where the ordering \prec is by inclusion and $(\lambda_0 < \cdots \hat{\lambda}_i \cdots < \lambda_{p+1}) = (\lambda_0 < \cdots < \lambda_{i-1} < \lambda_{i+1} \cdots < \lambda_{p+1})$. It is a ranked poset with minimal rank $m_{\Lambda^{(1)}} = 0$ of geometric dimension $d_{\Lambda^{(1)}} = r(\Lambda)$.

2.2 Projective systems and the projective limit

We start this section with a motivation for one more definition, using normal spline theory.

In spline theory, the work is done over a polynomial space, and give restrictions when the functions from two different spaces meet. They should overlap in some sence. Hence we want to assign to every element in a poset, a polynomial space. It will be natural, as in spline theory, to assign to every maximal cell the entire polynomial space and to the lower dimensional cells the polynomial space together with som restrictions. We could do it even more general to fit more cases, hence the definition given bellow, where we define to each cell an abelian group.

Definition 2.2.1. Let Λ be a poset, and let \mathcal{A} be the category of abelian groups (or any abelian category). A **projective system** with values in \mathcal{A} on Λ is a contravariant functor $F : \Lambda \rightarrow \mathcal{A}$, where Λ is considered a category, i.e. for any $\lambda \in \Lambda$ we associate an object $F(\lambda)$, and for a relation $\lambda < \sigma$, a morphism $F_{\lambda < \sigma} : F(\sigma) \rightarrow F(\lambda)$ in \mathcal{A} . Whenever, $\gamma \leq \tau \leq \sigma \in \Lambda$, we have $F_{\gamma < \tau} \circ F_{\tau < \sigma} = F_{\gamma < \sigma}$.

We are now ready to introduce one of the main tools in this thesis, namely the contravariant functor:

$$\varprojlim_{\Lambda/\Lambda_1} : \mathcal{C}_\Lambda \longrightarrow \mathcal{A}.$$

Our definition is taken from Laudal's article [1]:

Definition 2.2.2. Let Λ be a finite poset, and Λ_1 a closed subposet of Λ . Let \mathcal{C}_Λ be the abelian category of projective systems on Λ with values in the category \mathcal{A} of abelian groups. Then the category \mathcal{C}_Λ has enough injectives and projectives. For any projective system F , **the projective limit**

$$\varprojlim_{\Lambda/\Lambda_1} F$$

is an abelian group, together with a family of group homomorphisms

$$\Pi_\lambda : \varprojlim_{\Lambda/\Lambda_1} F \longrightarrow F(\lambda) .$$

It is unique, up to isomorphism, and universal with the above property. The homomorphisms define a natural transformation $\varprojlim_{\Lambda/\Lambda_1} F \longrightarrow F$, of the constant projective system $\varprojlim_{\Lambda/\Lambda_1} F$ into F and for all $\lambda \in \Lambda_1$, we have $\Pi_\lambda = 0$.

In other words it is such that for all $\lambda_j < \lambda_i$ the following diagram commutes:

$$\begin{array}{ccc} & \varprojlim_{\Lambda/\Lambda_1} F & \\ \Pi_{\lambda_i} \swarrow & & \searrow \Pi_{\lambda_j} \\ F(\lambda_i) & \xrightarrow{F_{\lambda_j < \lambda_i}} & F(\lambda_j) \end{array}$$

It is universal (in the means of universally repelling), which means that if (B, Φ_{λ_i}) is any object in the above category, then there exists a unique morphism $\phi : \varprojlim_{\Lambda/\Lambda_1} F \rightarrow B$ which makes the following diagram commutative:

$$\begin{array}{ccc} & B & \\ & \downarrow \phi & \\ & \varprojlim_{\Lambda/\Lambda_1} F & \\ \Phi_{\lambda_i} \swarrow & & \searrow \Phi_{\lambda_j} \\ \Pi_{\lambda_i} \swarrow & & \searrow \Pi_{\lambda_j} \\ F(\lambda_i) & \xrightarrow{F_{\lambda_j < \lambda_i}} & F(\lambda_j) \end{array}$$

It is shown in [1] that the limit exist and that the functor $\varprojlim_{\Lambda/\Lambda_1}$ is left exact.³

Thus an exact sequence

$$0 \rightarrow F'' \rightarrow F \rightarrow F' \rightarrow 0$$

of projective systems of abelian groups on Λ induces a long-exact sequence of abelian groups [11]:

$$\dots \rightarrow \varprojlim_{\Lambda/\Lambda_1}^{(j)} F'' \rightarrow \varprojlim_{\Lambda/\Lambda_1}^{(j)} F \rightarrow \varprojlim_{\Lambda/\Lambda_1}^{(j)} F' \rightarrow \varprojlim_{\Lambda/\Lambda_1}^{(j+1)} F'' \rightarrow \dots$$

for $j \geq 0$ where $\varprojlim_{\Lambda/\Lambda_1}^{(j)} F$ is the j -th right derived functor of $\varprojlim_{\Lambda/\Lambda_1} F$. See section 2.5 in Weibel [12] for further reading.

2.3 (Co-)homology

In section 2.3.1 we are going to define one type of cohomology, namely simplicial cohomology of the order complex, then in section 2.3.2 we are going to take a look on cellular (co-)homology. At last in section 2.3.3 we will see how the different (co-)homologies agrees. An example of the agreement and an illustration of the convenience of using cellular (co-)homology in some cases can be found in Appendix A.

Let us however start by recalling the notion of a complex:

Definition 2.3.1. Let A be a ring. An **(open) complex** of A -modules, is a sequence of modules and homomorphisms $\{(E^i, d^i)\}$,

$$\dots \rightarrow E^{i-1} \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \xrightarrow{d^{i+1}} E^{i+2} \rightarrow \dots$$

such that for all i

$$d^i \circ d^{i+1} = 0,$$

where $i \in \mathbb{Z}$ and d^i maps E^i into E^{i+1} .

³**Notation remark:** If $\Lambda_1 = \emptyset$, we set $\varprojlim_{\Lambda/\emptyset} = \varprojlim_{\Lambda}$.

2.3.1 Simplicial cohomology of the order complex

One last definition is needed for the simplicial cohomology of the order complex to fall out:

Definition 2.3.2. Let Λ be a ranked poset and F a projective system with values in \mathcal{A} . Define ξ to be a map⁴ $\xi : \Lambda < \Lambda < \cdots < \Lambda \rightarrow \mathcal{A}$. Let $(\lambda_0 < \lambda_1 < \cdots < \lambda_p) \in \Lambda$, then $\xi(\lambda_0 < \lambda_1 < \cdots < \lambda_p) \in F(\lambda_0)$.

We are now ready to start introducing the homology, and cohomology theory needed in this thesis. Let us start by defining a complex:

$$D^p(\Lambda, F) := \prod_{\lambda_0 < \lambda_1 < \cdots < \lambda_p \in \Lambda} F(\lambda_0), \quad (2.1)$$

where \prod means (\prod, \times) /direct product. Hence $D^p(\Lambda, F)$ is a product of "chains" in cell complexes taking the lowest value in each of its minimal cell.

The complex has differentials $\delta : D^p(\Lambda, F) \rightarrow D^{p+1}(\Lambda, F)$ given by

$$\begin{aligned} \delta \xi(\lambda_0 < \cdots < \lambda_{p+1}) &= F_{\lambda_0 < \lambda_1} [\xi(\lambda_1 < \cdots < \lambda_{p+1})] \\ &\quad + \sum_{i=1}^{p+1} (-1)^i \xi(\lambda_0 < \cdots < \hat{\lambda}_i \cdots < \lambda_{p+1}), \end{aligned}$$

where $(\lambda_0 < \cdots < \hat{\lambda}_i \cdots < \lambda_{p+1}) = (\lambda_0 < \cdots < \lambda_{i-1} < \lambda_{i+1} \cdots < \lambda_{p+1})$.

Definition 2.3.3. The cohomology groups of the complex $(D^\bullet(\Lambda, F^*), \delta)$,

$$H^p(D^\bullet(\Lambda, F)) = H^p(D^\bullet(\Lambda, F), \delta), \quad p \geq 0$$

are called the **order cohomology groups** of the ranked poset Λ with coefficients in the projective system F^* .

Further it is shown in [1] that the order cohomology groups constitute a resolving complex for the inverse limit functor, i.e. we have

Lemma 2.3.4.

$$H^p(D^\bullet(\Lambda, F)) = \varprojlim_{\Lambda}^{(p)} F, \quad p \geq 0$$

We now have our results and machinery on order cohomology of projective systems on ranked posets, so we give an example of how it may be used. First a notation remark:

⁴This map is not a function, since the range of the function varies for each of the elements in the domain.

Remark 2.3.5. Let A be a module, by abuse of notation, we will denote $\underbrace{A \times A \times \cdots \times A}_r$ as rA in the rest of the thesis.

We illustrate what the complex from equation (2.1) and its differentials would look like using a triangle and its poset as an example.

Example 2.3.6. Let Λ be the poset of a triangle, shown in the figure below.

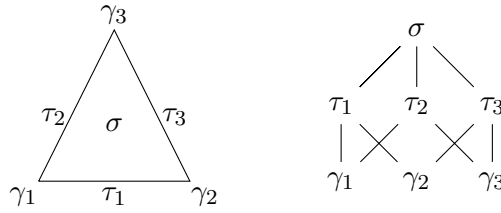


Figure 2.4: A triangle and its poset.

Then from the poset we compute:

$$\begin{aligned} D^0(\Lambda, F) &= \prod_{\lambda_0 \in \Lambda} F(\lambda_0) \\ &= F(\sigma) \times F(\tau_1) \times F(\tau_2) \times F(\tau_3) \times F(\gamma_1) \times F(\gamma_2) \times F(\gamma_3) \end{aligned}$$

$$\begin{aligned} D^1(\Lambda, F) &= \prod_{\lambda_0 < \lambda_1 \in \Lambda} F(\lambda_0) \\ &= \prod_{\substack{\gamma_i < \sigma \\ i=1,2,3}} F(\gamma_i) \times \prod_{\substack{\gamma_i < \tau_j \\ i,j=1,2,3}} F(\gamma_i) \times \prod_{\substack{\tau_i < \sigma \\ i=1,2,3}} F(\tau_i) \\ &= F(\tau_1) \times F(\tau_2) \times F(\tau_3) \times 3F(\gamma_1) \times 3F(\gamma_2) \times 3F(\gamma_3) \end{aligned}$$

$$\begin{aligned} D^2(\Lambda, F) &= \prod_{\lambda_0 < \lambda_1 < \lambda_2 \in \Lambda} F(\lambda_0) \\ &= \prod_{\substack{\gamma_1 < \tau_i < \sigma \\ i=1,2,3}} F(\gamma_1) \times \prod_{\substack{\gamma_2 < \tau_i < \sigma \\ i=1,2,3}} F(\gamma_2) \times \prod_{\substack{\gamma_3 < \tau_i < \sigma \\ i=1,2,3}} F(\gamma_3) \\ &= 2F(\gamma_1) \times 2F(\gamma_2) \times 2F(\gamma_3) \end{aligned}$$

$$\begin{aligned} D^3(\Lambda, F) &= \prod_{\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 \in \Lambda} F(\lambda_0) \\ &= 0 \end{aligned}$$

Further, the differentials are given by:

$$\begin{aligned}\delta^{0 \rightarrow 1} \xi(\lambda_0 < \lambda_1) &= F_{\lambda_0 < \lambda_1} [\xi(\lambda_1)] - \xi(\lambda_0) \\ \delta^{1 \rightarrow 2} \xi(\lambda_0 < \lambda_1 < \lambda_2) &= F_{\lambda_0 < \lambda_1} [\xi(\lambda_1 < \lambda_2)] - \xi(\lambda_0 < \lambda_2) + \xi(\lambda_0 < \lambda_1) \\ \delta^{2 \rightarrow 3} \xi(\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3) &= 0\end{aligned}$$

★

2.3.2 Cellular (co-)homology

Let Λ be a locally oriented ranked poset of geometric dimension d_Λ , with local orientation ϵ , and let F^* be a projective system of abelian groups on Λ^* . We define a cochain complex $C^p(\Lambda, F^*)$ to be:

$$C^p(\Lambda, F^*) = \prod_{\lambda \in \Lambda_p} F^*(\lambda^*),$$

with differential $\partial : C^p(\Lambda, F^*) \rightarrow C^{p+1}(\Lambda, F^*)$ given by

$$\partial \xi(\sigma) = \sum_{\lambda \in \Delta \sigma} \epsilon_{\lambda < \sigma} F_{\sigma^* < \lambda^*}^* \xi(\lambda)$$

By the local orientation property it follows that $\partial^2 = 0$. See page 3 [2] for the straight forward computational proof.

Definition 2.3.7. The cohomology groups of the complex $(C^\bullet(\Lambda, F^*), \partial)$,

$$H^p(\Lambda, F^*) = H^p(C^\bullet(\Lambda, F^*), \partial), \quad p \geq 0$$

are called the **cellular cohomology groups** of the locally oriented ranked poset Λ with coefficients in the projective system F^* .

In most situations it may be easier to use cellular homology instead of cellular cohomology, because then we will not need the dual poset, we give it a formal definition:

Let Λ be a locally oriented ranked poset of geometric dimension d_Λ , with local orientation ϵ , and let F be a projective system of abelian groups on Λ .

We define a chain complex:

$$C_p(\Lambda, F) = \coprod_{\lambda \in \Lambda_p} F(\lambda)[\lambda],$$

where \coprod means (\coprod, \oplus) /direct sum, with differential $\delta : C_p(\Lambda, F) \rightarrow C_{p-1}(\Lambda, F)$ given by

$$\delta(f_\sigma[\sigma]) = \sum_{\lambda \in \Delta \sigma} \epsilon_{\lambda < \sigma} F_{\lambda < \sigma}(f_\sigma)[\lambda]$$

for any $f_\sigma \in F(\sigma)$. Again we have $\delta^2 = 0$ by the local orientation property.

Definition 2.3.8. The homology groups of the complex $(C_\bullet(\Lambda, F), \delta)$,

$$H_p(\Lambda, F) = H_p(C_*(\Lambda, F)), \quad p \geq 0$$

are called the **cellular homology groups** of the locally oriented ranked poset Λ with coefficients in the projective system F .

Remark 2.3.9. Notice that equivalent local orientations give isomorphic cellular (co-)homology groups.

2.3.3 (Co-)homological relations

We need two results from Sletsjøe's manuscript [2], concerning the relations between cellular (co-)homology and order cohomology, to be able to compute the dimension of spline spaces using cellular (co-)homology.

Theorem 2.3.10. Let Λ be a locally oriented poset of geometric dimension $d = d_\Lambda$, with local orientation ϵ . Let F be a projective system on Λ . Then the cellular homology and cellular cohomology agrees, i.e.:

$$H^p(\Lambda^*, F) \simeq H_{d-p}(\Lambda, F), \quad p \geq 0$$

Proof. Proof given in [2], page 4. □

The next result is given as a corollary of theorem 3.3 in Sletsjøe's manuscript, we however give it as an own theorem. Since we will only need the corollary, not the theorem itself, we skip the restatement of his theorem. However, before we render the corollary we need one more definition:

Definition 2.3.11. A poset Λ is said to have acyclic closed cells if

$$H^p\left(\nabla^\bullet\left((\lambda^*)_q\right), F(\lambda_0)\right) = \begin{cases} F(\lambda_0) & \text{for } q = 0 \\ 0 & \text{for } q \neq 0 \end{cases}$$

Notice that we here are looking at the cellular cohomology of a constant poset, where every $\lambda_i \in \Lambda$ obtains the same value.

Theorem 2.3.12. Let Λ be a locally oriented combinatorial cell complex of geometrical dimension $d = d_\Lambda$, such that Λ^* has acyclic closed cells, and let F be a projective system of \mathcal{A} -modules on Λ . Then we have

$$H_p(\Lambda, F) = \varprojlim_{\Lambda}^{(d-p)} F, \quad p \geq 0$$

Proof. Proof given in [2], page 7. □

Corollary 2.3.13. Let Λ be a locally oriented combinatorial cell complex of geometrical dimension $d = d_\Lambda$, such that Λ^* has acyclic closed cells, and let F be a projective system of \mathcal{A} -modules on Λ . Then we have

$$H^p(\Lambda^*, F) = \varprojlim_{\Lambda^*} {}^{(p)}F, \quad p \geq 0$$

We desire a result saying that we in many of our cases could use cellular (co-)homology instead of order cohomology. This actually falls right out of the construction of cellular (co-)homology in the cases we are going to consider.

Proposition 2.3.14. Given a poset Λ , corresponding to a contractible plane area S , then S has acyclic closed cells. It follows that

$$H^p(C_*(\Lambda, F)) = H^{(d_\Lambda - p)}(D^\bullet(\Lambda, F), \delta) .$$

Proof. Follows directly from the fact that every poset corresponding to a contractible plane area S , has acyclic closed cells and from theorem 2.3.12. □

2.4 Spline systems

To be able to compute the dimension of a spline space, we first need to say something about how we define our projective systems. We will actually develop a special projective system, an approach we will use throughout this thesis. Of course we could have continued in the general setting, $\Delta \in \mathbb{R}$ and the projective system $F : \Lambda_\Delta \rightarrow \mathcal{A}$, but this would not be evenly interesting for people working with splines and spline spaces. Therefore we will proceed in a manner giving useful results, using an approach quite familiar to the "spline people". First:

What is the normal approach of splines and their definition?

Let us give the formal definition:

Definition 2.4.1. Let r be a positive integer and Δ a complex of polyhedra of a plane area S . The vector space $\mathbf{C}^r(\Delta)$ is the collection of C^r -functions f on Δ such that for every cell $\delta \in \Delta$ the restriction $f|_\delta$ is a polynomial function $f_\delta \in S$. For $n \in \mathbb{N}_0$, $\mathbf{S}_n^r(\Delta)$ is the subset of $\mathbf{C}^r(\Delta)$ such that the restriction of f to each cell $\delta \in \Delta$ is a polynomial of degree $\leq n$.

To get our theory like "theirs" we have to find a "good" projective system:

What type of projective systems are useful in "our spline setting"?

To be meaningful we would of course like our projective system to be a category where restriction maps are surjective. If this wasn't the case we would just have a lot of zeros in the category, which wouldn't make any sense. The other property would be that all the maximal cells are constants. We give a formal definition:

Definition 2.4.2. A projective system for which all restriction maps are surjective is called a **surjective system**. A **spline system** is a surjective system which is constant on cells of maximal rank.

Every spline system has a corresponding kernel system. However a kernel system is not a spline system, because the mappings from a cell to a lower dimension cell is not surjective, in fact they happen to be injective.⁵

Let us recall the notion of a **filtration**:

Definition 2.4.3. Let A be an algebra over a field k . By a **filtration** of A , we mean a sequence of k -vector spaces A_i , $i = 0, 1, \dots$ such that

$$A_0 \subset A_1 \subset A_2 \subset \dots, \bigcup_i A_i = A,$$

and $A_i A_j \subset A_{i+j}$ for all $i, j \geq 0$.

We are now ready to introduce our approach:

Approach 2.4.4. To any complex Δ we associate a poset Λ where cells are nodes and relations are given by incidence. On the poset Λ_Δ we define a projective system F of polynomial functions, i.e. a contravariant functor $F : \Lambda_\Delta \rightarrow k\text{-vector space}$.

There is a spline system on Λ_Δ such that for any 2-cell σ , $F(\sigma) = R = k[x, y]_{\leq n}$, the vector space of polynomial functions in two variable, of degree less or equal to n . A 1-cell $\tau \in \Lambda$ corresponds to the zero set of a linear functional L_τ , and we let

$$F(\tau) = (k[x, y]/(L_\tau^{r+1}))_{\leq n}.$$

⁵It may be tempting to call them injective systems, this however is not a good idea. Since while we are working with projective systems an injective system corresponds to a projective system by just turning the maps. If we should have given it a reasonable name it could have been a **system of injective maps**.

For a vertex $\gamma \in \Lambda$ we define

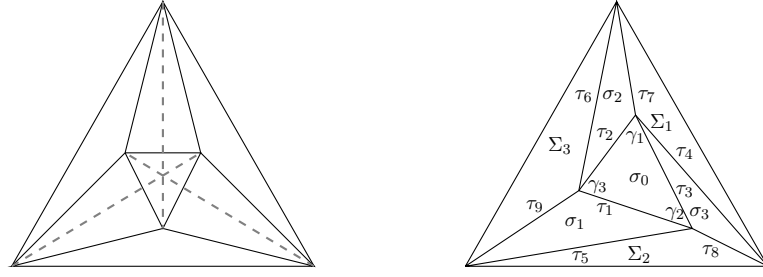
$$I_\gamma = \bigcap_{\tau \ni \gamma} (L_\tau)^{r+1},$$

and

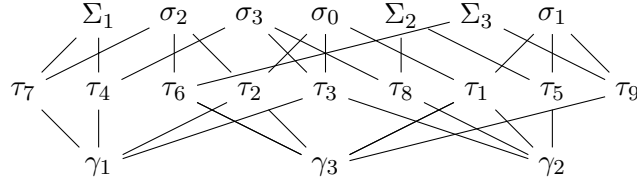
$$F(\gamma) = (k[x, y]/I_\gamma)_{\leq n}.$$

Notice that F is well-defined in the approach since $k[x, y]$ is a filtered ring and $I_\tau = (L_\tau)^{r+1}$ and I_γ are filtered ideals in $k[x, y]$. Hence the approach defines a projective system of truncated filtered k -algebras on Δ .

Example 2.4.5. An example of a spline system like this special spline system F , can be shown by letting Δ_{MS} be the Morgan-Scott triangulation, shown below.

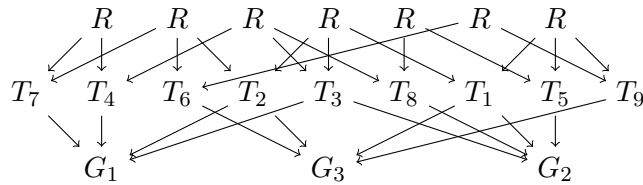


The Morgan-Scott triangulation Δ , has 7 2-cells, denoted by Σ_i , $i = 1, 2, 3$, and σ_j , $j = 0, 1, 2, 3$, depending on whether it has a boundary facet or not, while the inner 1-cells are denoted by τ_k , $k = 1, \dots, 9$ and the three inner vertices by γ_l , $l = 1, 2, 3$. Its poset is shown below.



How does the spline system of Δ_{MS} look like?

We see its spline system below, letting $T_k = F(\tau_k) = (k[x, y]/(L_{\tau_k}^{r+1}))_{\leq n}$, $G_l = F(\gamma_l) = (k[x, y]/I_{\gamma_l})_{\leq n}$ and \rightarrow be surjective maps:



We will consider the dimension of the Morgan-Scott triangulation closer in section 4.2. ★

Remark 2.4.6. Throughout this thesis we will use the approach given in 2.4.4 for all spline systems denoted by F .

In the illustrations of the spline systems, the surjective maps from $F(\sigma) \mapsto F(\tau)$, $\tau < \sigma \in \Lambda$, will not be given as \rightarrow in the previous page, but as an relation by $-$.

2.5 The dimension of a spline space

We have now established the theoretical environment for the thesis and are ready to approach our main goal of finding the dimension of different spline spaces. In the former sections we have seen the relation between a poset and the geometric grid often used in spline spaces. We are almost ready to give the formal relation between the projective limit over a poset and the dimension of its corresponding spline space.

However we need to agree on what the dimension of an ideal I is. Our agreement will be:

Let $R = k[x, y]$. We denote by

$$r_k = \dim_k(k[x, y]_k).$$

Then $r_k = k + 1$, $\dim R = \binom{n+2}{2}$ and

$$r_{[k,m]} = r_k + r_{k+1} + \cdots + r_m = \binom{n+2}{2} - \binom{r+1}{2}.$$

Definition 2.5.1. The **dimension of an ideal** I is said to be the dimension of an ideal as a vector space, hence

$$\dim I = \dim(L^{r+1}) \cap R_{\leq n} = \binom{n+1-r}{2}.$$

Before we see the agreement of $\varprojlim_{\Lambda_\Delta} F$ and $S_n^r(\Delta)$ we would like to establish the following:

Lemma 2.5.2. Let f be a piecewise polynomial function over a triangulation Δ of a plane region S , with almost everywhere continuous k -th derivative. Let σ_1 and σ_2 be two adjacent 2-cells, and $\tau \subset \sigma_1 \cap \sigma_2$ a 1-cell defined by a linear functional L_τ . Then $f(\sigma_1) - f(\sigma_2) \in (L_\tau)^{r+1}$ and vice versa.

Proof. Let $x \in \tau$ be different from the endpoints of τ . Then $D^k f$ is continuous in x if and only if $D^k(f(\sigma_1) - f(\sigma_2))(x) = 0$. This is equivalent to $f(\sigma_1) - f(\sigma_2) \in (L_\tau)^{r+1}$. □

Proposition 2.5.3. Let F be our spline system where $F(\sigma) = k[x, y]_{\leq n}$ and $F(\tau) = (k[x, y]/(L_\tau^{r+1}))_{\leq n}$, for $\sigma \in \Lambda_{m_\Lambda}$ and $\tau \in \Lambda_{(m_\Lambda-1)}$, and let $S_n^r(\Delta)$ the spline space over Δ . Then:

$$S_n^r(\Delta) \simeq \varprojlim_{\Lambda_\Delta} F.$$

Proof. The result follows from the nature of $\varprojlim_{\Lambda_\Delta} F$. Because $\varprojlim_{\Lambda_\Delta} F$ comes from all the maximum cells, and is such that it fits together in all $(m_\Lambda - 1)$ -cells. I.e.

$$\varprojlim_{\Lambda_\Delta} F : \prod_{\lambda_0 \in \Lambda} F(\lambda_0) \rightarrow \prod_{\lambda_0 < \lambda_1 \in \Lambda} F(\lambda_0).$$

Take $\xi \in \prod_{\lambda_0 \in \Lambda} F(\lambda_0)$ and $\sigma_1, \sigma_2 \in \Lambda$ such that $\Delta\sigma_1 \cap \Delta\sigma_2 = \tau$, then we have:

$$\begin{aligned} \delta\xi(\tau < \sigma_1) = 0 & \Rightarrow p\xi(\sigma_1) = \xi(\tau) \\ \delta\xi(\tau < \sigma_2) = 0 & \Rightarrow p\xi(\sigma_2) = \xi(\tau), \end{aligned}$$

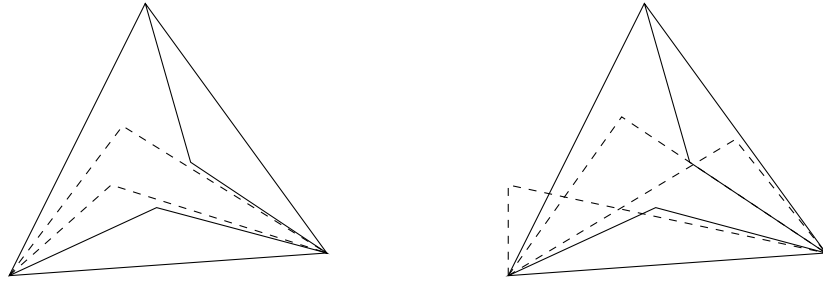
where p is the projection of $\xi(\sigma)$ to τ . Hence $p\xi(\sigma_1) = p\xi(\sigma_2)$, which is exactly the same as $S_n^r(\Delta)$. \square

The following observation will come in handy later on:

Observation 2.5.4. The dimension of $S_n^r(\Delta)$ stays unaffected during any movement of the edges, as long as the number of points where the edges coincide with each other remain constant.

To get a better understanding of what is meant by the observation above, we illustrate it by an example:

Example 2.5.5. Consider the continuous line triangulation below:



The left figure illustrates movements of the edges which doesn't affect the dimension, while the right figure illustrates three types of edge-movements which may affect the dimension. \star

We will need the following result in section 4.2.1, but it may also come in handy when computing dimension of small subspaces.

Proposition 2.5.6. Given two ranked posets Λ^* and Λ , such that Λ^* has only one minimal element, γ , and $\Lambda = \Lambda^* \setminus \gamma$. Define a projective system F^* on Λ^* such that $F^*(\gamma) = 0$ and elsewhere equal to F . Then:

$$\varprojlim_{\Lambda^*} F^* \simeq \varprojlim_{\Lambda} F.$$

Proof. Follows by the construction of \varprojlim . □

Finally in this chapter we give two results. The first one will be used in almost every proof in the rest of the thesis, when it comes to computing \varprojlim_{Λ} of a kernel system.

Lemma 2.5.7. For a projective system F on a poset Λ such that F vanishes on all maximal elements, we have

$$\varprojlim_{\Lambda} F = 0.$$

Proof. The proof is straight forward and follows by the construction of $\varprojlim_{\Lambda} F$. □

Proposition 2.5.8. Let F be a quotient of a spline system on a simply-connected planar cell complex Δ . Then

$$\varprojlim_{\Lambda_{\Delta}}^{(2)} F = 0.$$

Proof. Let R be the constant projective system on Δ , such that there is a surjective map $R \rightarrow F$, with kernel K . Then we have

$$\cdots \rightarrow \varprojlim_{\Lambda_{\Delta}}^{(2)} R \longrightarrow \varprojlim_{\Lambda_{\Delta}}^{(2)} F \longrightarrow \varprojlim_{\Lambda_{\Delta}}^{(3)} K \rightarrow \cdots.$$

But for a constant system on a simply-connected planar cell complex, the higher derivatives of the inverse limit functor vanish. Also the third derivative of a projective system on poset of geometric dimension 2 vanishes. Hence $\varprojlim_{\Lambda_{\Delta}}^{(2)} F = 0$. □

Chapter 3

Main Tools for Computing the Dimension of Spline Spaces

This chapter will illustrate how we may compute the dimension of a spline space, using the tools established in chapter 2. However if the geometric complex Δ of the spline space are defined over are "too large", it is nearly impossible to use those tools alone. We will therefore in chapter 4 introduce a new tool to *degenerate* a spline space into some of the geometric complexes presented in this section.

In the following section we construct tools to simplify computing of the dimension of spline spaces presented later in this chapter.

3.1 Basic shapes and operations

The first proposition states that if an edge τ in a ranked poset Λ doesn't split a 2-cell, τ could have been omitted from the poset without a change in $\varprojlim_{\Lambda} F$.

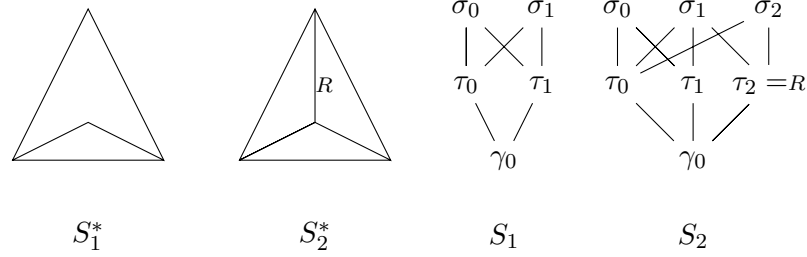
Proposition 3.1.1. Given two ranked posets Λ_1 and Λ_2 , with $\Lambda_1 \setminus \tau = \Lambda_2$, for a $\tau \in \Lambda_1$. And given a spline system F' on Λ_1 equal to F on Λ_2 , but $F'(\tau) = R$, then:

$$\varprojlim_{\Lambda_1} F' \simeq \varprojlim_{\Lambda_2} F.$$

Proof. Follows by the construction of \varprojlim_{Λ} . However because of proposition 2.5.3 we could have argued this way: Since the spline space of Λ_2 is equal to $\varprojlim_{\Lambda_2} F$, it follows from spline theory that we may add an edge without any restrictions, but still not changing the dimension of the spline space. \square

To make things clearer we illustrate proposition 3.1.1 by an example:

Example 3.1.2. Consider the planar object S_1^* shown in the figure below.



By adding an edge $F(\tau_2) = R$, we get S_2^* . By the poset to the right we observe that since σ_0 and σ_2 should be equal everywhere ($F(\tau_2) = R$), we have to have $\sigma_0 = \sigma_2$. ★

The following lemma will be useful in many cases when we compute order cohomology and need to simplify the kernel complex by *decomposing* it. In those cases we would like to ensure that we have a commutative diagram, hence we give a general argument for all of those cases.

Lemma 3.1.3. Given a partially ordered set Λ , a closed subset $\Lambda_1 \in \Lambda$, a spline system F , with differentials $\delta : D^p(\Lambda, F) \rightarrow D^{p+1}(\Lambda, F)$ and a restriction map $\iota : D^p(\Lambda, F) \rightarrow D^p(\Lambda_1, F) \forall p$. Then $\iota \circ \delta = \delta \circ \iota$.

Proof. Let $\lambda_0 < \dots < \lambda_{p+1} \in \Lambda_1$ then

$$\iota \delta \psi(\lambda_0 < \dots < \lambda_{p+1}) = \delta \psi(\lambda_0 < \dots < \lambda_{p+1})$$

and

$$\delta \iota \psi(\lambda_0 < \dots < \lambda_{p+1}) = \delta \psi(\lambda_0 < \dots < \lambda_{p+1})$$

since $\iota \psi(\underline{\lambda}) = \psi(\underline{\lambda})$ when $\underline{\lambda} \in \Lambda_1$. Hence $\iota \circ \delta = \delta \circ \iota$. □

We will now establish the known fact that the spline space of the planar object is the same independent of the inclusion of its boundary. We will do this by illustrating how the spline space over planar object can be computed by our methods.

After this proposition we will omit the boundary of all planar complexes for the rest of the thesis.

Proposition 3.1.4. Given a planar complex Δ , the spline space $S_n^r(\Delta)$ is the same both with and without the boundary (of Δ) included in Δ .

Proof. The planar object Δ has an associated ranked poset, denoted Λ . Now let F be our spline system given in the approach 2.4.4 and let F_0 be a projective system on Λ such that for all elements $\lambda \in \Lambda$, $F_0(\lambda) = F(\lambda)$, except for the elements λ' on the boundary of Λ , where $F(\lambda') = R$.

Without lose of generality we may set the entier interior of $\Lambda = \sigma$. The shape of the polygon making the boundary does not effect \varprojlim_{Λ} , hence we can assume that Δ equals a triangle.

Now construct the short exact sequence, by letting K be the kernel spline system of F_0 and F :

$$0 \rightarrow K \rightarrow \begin{array}{c} \triangle \\ \sigma \\ F_0 \end{array} \rightarrow \begin{array}{c} \gamma_{23} \\ \triangle \\ \tau_2 \quad \sigma \quad \tau_3 \\ \gamma_{12} \quad \tau_1 \quad \gamma_{13} \\ F \end{array} \rightarrow 0$$

The spline systems are given by:

$$0 \rightarrow \begin{array}{c} 0 \\ \swarrow \quad | \quad \searrow \\ I_1 \quad I_2 \quad I_3 \\ | \quad \times \quad \times \quad | \\ I_1 + I_2 \quad I_1 + I_3 \quad I_2 + I_3 \end{array} \rightarrow \begin{array}{c} R \\ \swarrow \quad | \quad \searrow \\ R \quad R \quad R \\ | \quad \times \quad \times \quad | \\ R \quad R \quad R \end{array} \rightarrow \begin{array}{c} R \\ \swarrow \quad | \quad \searrow \\ R/I_1 \quad R/I_2 \quad R/I_3 \\ | \quad \times \quad \times \quad | \\ R/(I_1 + I_2) \quad R/(I_1 + I_3) \quad R/(I_2 + I_3) \end{array} \rightarrow 0$$

To see that there is no difference we need to compute $\varprojlim_{\Lambda}^{(1)} K$ by computing H^1 of the order cohomology complex of K :

$$\begin{aligned} & I_1 \times I_2 \times I_3 \times (I_1 + I_2) \times (I_1 + I_3) \times (I_2 + I_3) \\ & \quad \downarrow \delta_K^0 \\ & I_1 \times I_2 \times I_3 \times 3(I_1 + I_2) \times 3(I_1 + I_3) \times 3(I_2 + I_3) \\ & \quad \downarrow \delta_K^1 \\ & 2(I_1 + I_2) \times 2(I_1 + I_3) \times 2(I_2 + I_3) \end{aligned}$$

We construct a diagram by projecting down to the vertices:

$$\begin{array}{ccccccc}
0 & & & & & & \\
\downarrow & & & & & & \\
I_1 \times I_2 \times I_3 & \xleftarrow{\delta_{\ker}^0} & I_1 \times I_2 \times I_3 & \xrightarrow{\delta_{\ker}^1} & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
I_1 \times I_2 \times I_3 \times (I_1 + I_2) \times (I_1 + I_3) \times (I_2 + I_3) & \xrightarrow{\delta_K^0} & I_1 \times I_2 \times I_3 \times 3(I_1 + I_2) \times 3(I_1 + I_3) \times 3(I_2 + I_3) & \xrightarrow{\delta_K^1} & 2(I_1 + I_2) \times 2(I_1 + I_3) \times 2(I_2 + I_3) & & \\
\downarrow p & & \downarrow p & & \downarrow p & & \\
(I_1 + I_2) \times (I_1 + I_3) \times (I_2 + I_3) & \xleftarrow{\delta_p^0} & 3(I_1 + I_2) \times 3(I_1 + I_3) \times 3(I_2 + I_3) & \xrightarrow{\delta_p^1} & 2(I_1 + I_2) \times 2(I_1 + I_3) \times 2(I_2 + I_3) & &
\end{array}$$

By lemma 3.1.3, we verify that the diagram is commutative ($p \circ \delta = \delta \circ p$).

Now, to compute $H^1(K) = \ker d^1 / \text{Im } d^0$, we observe that all of the three rows in the diagram are exact, hence we do not have any cohomology groups. I.e. $H^1(K) = 0 = \varprojlim_{\Delta}^{(1)} K$ and by the long exact sequence:

$$\begin{array}{ccccccc}
\varprojlim_{\Lambda} K & \rightarrow & \varprojlim_{\Lambda} F_0 & \rightarrow & \varprojlim_{\Lambda} F & \rightarrow & \varprojlim_{\Lambda}^{(1)} K \rightarrow \varprojlim_{\Lambda}^{(1)} F_0 \rightarrow \dots \\
\parallel & & & & \parallel & & \parallel \\
0 & & & & 0 & & 0
\end{array}$$

where $\varprojlim_{\Lambda} K = 0$ by lemma 2.5.7, $\varprojlim_{\Lambda} F_0 \simeq \varprojlim_{\Lambda} F$. Hence the spline space is the same boundary included or not. \square

Lemma 3.1.5. Given a single planar object Δ , together with its poset $\Lambda_{\Delta} = \sigma$. Adding an edge τ to Λ_{Δ} , such that $F(\tau) = R/I$, inside Δ gives:

$$\varprojlim_{\Lambda_{\Delta}^*} F \simeq I \oplus \varprojlim_{\Lambda_{\Delta}} F,$$

where Δ^* is Δ after adding τ .

Proof. The boundary of Δ does not give any contributions, and hence we may illustrate Δ as a square.

We start by making an exact sequence of spline systems:

$$0 \rightarrow K \rightarrow \begin{array}{|c|} \hline \square \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \rightarrow 0$$

$F_0 \qquad F$

where F_0 is the projective system given by $F_0(\tau) = R$, and equal to F elsewhere. K is the kernel of the spline systems.

$$0 \rightarrow K \rightarrow \begin{array}{|c|} \hline R \\ \hline R \\ \hline R \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline R \\ \hline R/I \\ \hline R \\ \hline \end{array} \rightarrow 0$$

The spline systems are fully written:

$$0 \rightarrow \begin{array}{c} 0 \quad 0 \\ \backslash \quad / \\ I \end{array} \rightarrow \begin{array}{c} R \quad R \\ \backslash \quad / \\ R \end{array} \rightarrow \begin{array}{c} R \quad R \\ \backslash \quad / \\ R/I \end{array} \rightarrow 0$$

and the order cohomology complex of the kernel gives :

$$I \xrightarrow{\delta_K^0} 2I$$

Hence $H^1(K) = \varprojlim_{\Lambda_{\Delta^*}}^{(1)} K = I$ and the long exact sequence of the complexes shows that the difference between $\varprojlim_{\Lambda_{\Delta^*}} F$ and $\varprojlim_{\Lambda_{\Delta^*}} F_0 \simeq \varprojlim_{\Lambda_{\Delta}} F$ is I :

$$\begin{array}{ccccccccc} 0 & \rightarrow & \varprojlim_{\Lambda_{\Delta}} F & \rightarrow & \varprojlim_{\Lambda_{\Delta^*}} F & \rightarrow & \varprojlim_{\Lambda_{\Delta^*}}^{(1)} K & \rightarrow & \varprojlim_{\Lambda_{\Delta}}^{(1)} F & \rightarrow & \varprojlim_{\Lambda_{\Delta^*}}^{(1)} F & \rightarrow & 0 \\ & & \parallel & & & & \parallel & & \parallel & & \parallel & & \\ & & R & & & & I & & 0 & & 0 & & \end{array}$$

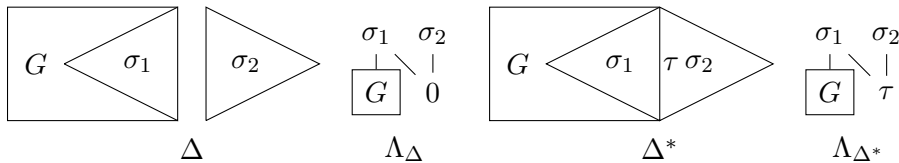
Observe also that $H^1(F) = 0$ by the fact that its order complex is surjective in δ^1 . \square

Corollary 3.1.6. Given a complex G , two cells σ_1 and $\sigma_2 \in G$, such that $\Delta^1 \sigma_1 \cap \Delta^1 \sigma_2 = \tau$ and $F(\tau) = R/I$. *Melting* together σ_1 with σ_2 by removing the edge τ gives a loss in dimension of G corresponding to the dimension of the ideal I .

Proposition 3.1.7. Let Δ be a planar object containing a two objects G and σ_2 : G a planar object G with a 2-cell σ_1 on the boundary and σ_2 a 2-cell such that $\sigma_2 \cap G = \emptyset$. Further given our usual spline system F on Δ .

Gluing the two disjoint cells σ_1 and σ_2 together through τ , achieving a new planar object Δ^* gives $\varprojlim_{\Delta} F \simeq \varprojlim_{\Delta^*} F \oplus F(\tau) \simeq \varprojlim_{\Delta^*} F \oplus R/I$.

Proof. The situation is illustrated in figure below, where Λ_{Δ} and Λ_{Δ^*} are the posets of Δ and Δ^* respectively.



It is obvious that

$$\varprojlim_{\Lambda_\Delta} F \simeq \varprojlim_G F \oplus \varprojlim_{\sigma_2} F \simeq \varprojlim_G F \oplus R,$$

and

$$\varprojlim_{\Lambda_{\Delta^*}} F \simeq \varprojlim_G F \oplus I,$$

so it is given by

$$\varprojlim_{\Lambda_\Delta} F \simeq \varprojlim_{\Lambda_{\Delta^*}} F \oplus R/I.$$

□

3.2 Planar spline stars

We will now look at and compute the dimension of a general structure appearing in all posets, as subposets, namely the planar spline stars. They may of course be generalized to higher dimensions where they are denoted as n -dimensional spline stars.

3.2.1 Open spline stars

Definition 3.2.1. The edges of a cell are said to be in **general position** if they do not coincide (in any meaning of the word).

Definition 3.2.2. An **open spline star** is a spline system whose poset is on the form shown in figure 3.1.

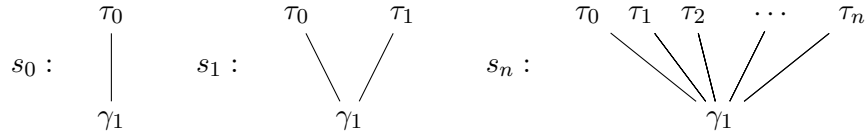


Figure 3.1: The posets s_0 , s_1 and s_n .

They are called open spline stars because of their geometric realization, shown in figure 3.2.



Figure 3.2: Geometric realizations s_0^* , s_1^* and s_n^* .

We let s_n denote the poset of the open spline star of degree n and denote its geometric realization by s_n^* .

Proposition 3.2.3. Let s_1 be the open spline star of degree 1, "defined" in figure 3.1. Then we have

$$\varprojlim_{s_1} F = F(\tau_0) \times_{F(\gamma)} F(\tau_1).$$

Further since $F(\tau_0) = R/I_0$, $F(\tau_1) = R/I_1$ and $F(\gamma) = R/I_0 + I_1$, then we have

$$F(\tau_0) \times_{F(\gamma)} F(\tau_1) = R/I_0 \times_{R/(I_0+I_1)} R/I_1 \simeq R/I_0 \cap I_1.$$

The following proposition follows by induction:

Proposition 3.2.4. Let s_n be the open spline star of degree n , "defined" in figure 3.1, and let $F(\tau_i) = R/I_i$ for all $i \in [0, n]$. Then

$$\varprojlim_{s_n} F \simeq R / \bigcap_{i=0}^n I_i$$

Let us now consider the dimension of what we call **spline stars**.

3.2.2 Spline stars

Definition 3.2.5. The spline systems of a **spline star** is of the form shown in figure 3.3. As for the open spline stars we let n be the degree of the spline star. We denote the spline star of degree n by \mathbf{S}_n .

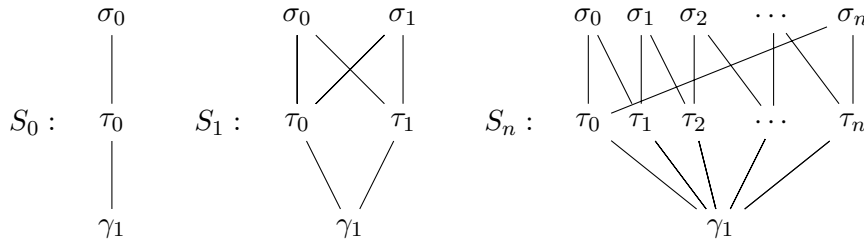


Figure 3.3: The posets S_0 , S_1 and S_n .

Remark 3.2.6. Notice that if we projected down on the 0- and 1-cells they are the same as the open spline stars.

We would now like to compute a formula for the dimension of a spline star of degree n . We start by considering the case $n = 1$:

$$\varprojlim_{S_1} F \simeq R \oplus (I_0 \cap I_1)$$

Proof. We start by making our short exact sequence of spline systems:

where F_0 is the constant spline system on S_1 and K is the kernel system of F_0 and F .

$$0 \xrightarrow{\delta_2} I_0 \oplus I_1 \xrightarrow{\delta_1} (I_0 + I_1),$$
$$\begin{aligned} H_1(K) &= \ker(I_0 \oplus I_1 \rightarrow (I_0 + I_1)) = I_0 \cap I_1 \\ H_0(K) &= \ker(\delta_0)/\operatorname{Im}(\delta_1) = 0/\operatorname{Im}(\delta_1) = 0. \end{aligned}$$

From our results in chapter 2 it follows that $H_i(K) = \varprojlim_{S_1}^{(2-i)} K$ and then from the long exact sequence:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \varprojlim_{S_1} K & \longrightarrow & \varprojlim_{S_1} F_0 & \longrightarrow & \varprojlim_{S_1} F \longrightarrow \varprojlim_{S_1}^{(1)} K \\
& & \parallel & & \parallel & & \parallel \\
& & 0 & & R & & I_0 \cap I_1 \\
& & & & & & \\
& & & & & & \longrightarrow \varprojlim_{S_1}^{(1)} F \longrightarrow \varprojlim_{S_1}^{(1)} F \longrightarrow \varprojlim_{S_1}^{(2)} K \\
& & & & & & \parallel \\
& & & & & & 0
\end{array}$$

where $\varprojlim_{S_1} K = 0$, by lemma 2.5.7 it follows that

$$\varprojlim_{S_1} F \simeq \varprojlim_{S_1} F_0 \oplus \varprojlim_{S_1}^{(1)} K \simeq R \oplus (I_0 \cap I_1),$$

and that $\varprojlim_{S_1}^{(1)} F \simeq 0$. Hence $\varprojlim_{S_1}^{(j)} F \simeq 0$ for all $j \geq 1$, since from proposition 2.5.8 it follows that $\varprojlim_{S_1}^{(k)} F \simeq 0$ for all $k \geq 2$. \square

Theorem 3.2.8. Given a spline star, S_n , of degree $n \geq 1$ where all the $n+1$ edges, τ_i , are in general position and given our usual spline system F on S_n , then:

$$\varprojlim_{S_n} F \simeq R \oplus \sum_{i=1}^n \left(I_i \cap \sum_{j=0}^{i-1} I_j \right),$$

and $\varprojlim_{S_n}^{(j)} F \simeq 0$ for all $j \geq 1$, where I_i , $i = 0, \dots, n$, are the ideals, corresponding to $F(\tau_i)$.

Proof. We do an inductive proof. From proposition 3.2.7 we know that

$$\varprojlim_{S_1} F \simeq R \oplus (I_0 \cap I_1) \quad \text{and} \quad \varprojlim_{S_1}^{(j)} F \simeq 0 \quad \forall j \geq 0.$$

So our statment holds for $n = 1$.

Now we assume the statement holds for all indexes up to and including $k-1$, i.e.:

$$\varprojlim_{S_{k-1}} F \simeq R \oplus \sum_{i=1}^{k-1} \left(I_i \cap \sum_{j=0}^{i-1} I_j \right) \quad \text{and} \quad \varprojlim_{S_{k-1}}^{(j)} F \simeq 0 \quad \forall j \geq 1.$$

We want to show that it then holds also for k , i.e. that

$$\varprojlim_{S_k} F \simeq R \oplus \sum_{i=1}^k \left(I_i \cap \sum_{j=0}^{i-1} I_j \right) \quad \text{and} \quad \varprojlim_{S_k}^{(j)} F \simeq 0 \quad \forall j \geq 1.$$

Or equivalently for the first statement

$$\varprojlim_{S_k} F \simeq \varprojlim_{S_{k-1}} F \oplus I_k \cap \left(\sum_{i=0}^{k-1} I_i \right).$$

Our usual starting point is the short exact sequence of spline systems. We let F_0 be the spline system given by $F_0(\tau_k) = R$ and equal to F elsewhere. Hence $\varprojlim_{S_k} F \simeq \varprojlim_{S_k} F_0$. We are now ready to start:

$$0 \rightarrow \begin{array}{c} \text{Diagram of } K \\ \text{Kernel spline system } K \end{array} \rightarrow \begin{array}{c} \text{Diagram of } F_0 \\ \text{Spline system } F_0 \end{array} \rightarrow \begin{array}{c} \text{Diagram of } F \\ \text{Spline system } F \end{array} \rightarrow 0$$

For the kernel spline system K all of the 1-cells are zero except from the $(k+1)^{\text{th}}$ 1-cell, which is I_k , the 0-cell is $J_0 = \left(\sum_{i=0}^k I_i \right) / \left(\sum_{i=0}^{k-1} I_i \right) \simeq I_k / \left(I_k \cap \sum_{i=0}^{k-1} I_i \right)$ and all the 2-cells are 0.

$$0 \rightarrow \begin{array}{c} \text{Diagram of } J_0 \\ \text{0-cell } J_0 \end{array} \rightarrow \begin{array}{c} \text{Diagram of } R / \sum_{i=0}^{k-1} I_i \\ \text{1-cell } R / \sum_{i=0}^{k-1} I_i \end{array} \rightarrow \begin{array}{c} \text{Diagram of } R / \sum_{i=0}^k I_i \\ \text{1-cell } R / \sum_{i=0}^k I_i \end{array} \rightarrow 0$$

To see the difference between $\varprojlim_{S_k} F$ and $\varprojlim_{S_k} F_0 \simeq \varprojlim_{S_{k-1}} F$, we need to compute $\varprojlim_{S_k}^{(1)} K$, hence we examine the cellular homology complex of K :

$$0 \xrightarrow{\delta_2} I_k \xrightarrow{\delta_1} J_0.$$

We compute the needed homology groups:

$$H_1(K) = \ker(I_k \rightarrow J_0) = \ker \left(I_k \rightarrow I_k / \left(I_k \cap \sum_{i=0}^{k-1} I_i \right) \right) = I_k \cap \sum_{i=0}^{k-1} I_i$$

$$H_0(K) = \ker(\delta_0) / \operatorname{Im}(\delta_1) = 0 / \operatorname{Im}(\delta_1) = 0$$

From theorem 2.3.12 it follows that $H_i(K) = \varprojlim_{S_k}^{(2-i)} K$ and then from the long exact sequence:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \varprojlim_{S_{k-1}} F & \longrightarrow & \varprojlim_{S_k} F & \longrightarrow & \varprojlim_{S_k}^{(1)} K \\
& & & & \parallel & & \\
& & & & I_k \cap \sum_{i=0}^{k-1} I_i & & \\
& & & & & & \\
& & & & \longrightarrow & \varprojlim_{S_{k-1}}^{(1)} F & \longrightarrow \varprojlim_{S_k}^{(1)} F \longrightarrow \varprojlim_{S_k}^{(2)} K \\
& & & & \parallel & & \parallel \\
& & & & 0 & & 0
\end{array}$$

it follows that

$$\varprojlim_{S_{k-1}} F \oplus I_k \cap \left(\sum_{i=0}^{k-1} I_i \right) \simeq \varprojlim_{S_k} F \quad \text{and} \quad \varprojlim_{S_{k-1}}^{(1)} F \simeq 0.$$

$\varprojlim_{S_{k-1}}^{(j)} F \simeq 0$ will hold for all $j \geq 1$, since it from proposition 2.5.8 holds for all $j \geq 2$. Hence we have proven that our statements hold for k if they hold for $k-1$.

So by induction from $k=1$ it follows that:

$$\varprojlim_{S_n} F \simeq R \oplus \sum_{i=1}^n \left(I_i \cap \sum_{j=0}^{i-1} I_j \right) \quad \text{and} \quad \varprojlim_{S_n}^{(j)} F \simeq 0 \quad \forall j \geq 1.$$

for all $n \geq 1$. □

3.3 T-splines

One of the most frequently problems discussed in spline theory are T-meshes and their dimension. In particular Deng, Chen and Feng discussed in [5] the problem of deciding the dimension for T-meshes. We will have a look at it.

Definition 3.3.1. Given a system with two lines intersecting in one point, such that their complex contains the structure T shown in figure 3.4, we call it a **T-junction**. A system containing T-junctions is called a **T-mesh**.

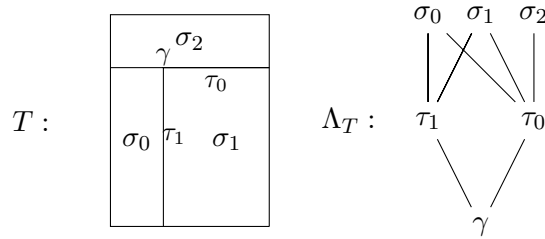


Figure 3.4: A system containing a T-junction and its poset.

Notice however that τ_0 actually consists of two edges: $(\text{boarder}, \gamma)$ and $(\gamma, \text{boarder})$, however since their projective systems coincide we have assigned them as τ_0 .

Lemma 3.3.2. Let T be the planar object with a T-junction illustrated in figure 3.4. Then:

$$\varprojlim_{\Lambda_T} F \simeq R \oplus I_0 \oplus I_1 / (I_0 \cap I_1)$$

where I_0 and I_1 corresponds to τ_0 and τ_1 , respectively.

Proof. Using the notation in lemma 3.1.5, $\Lambda_\Delta^* = s_1$ and $\Lambda_\Delta = \sigma$ and $F(\tau_0) = R/I_0$ for $\tau_0 \in s_1$, we know that $\varprojlim_{\Lambda_\Delta^*} F \simeq I_0 \oplus \varprojlim_{\Lambda_\Delta} F$.

Hence it suffices to prove that adding τ_1 to the poset s_1 , such that we get the poset Λ_T in figure 3.4, gives the short exact sequence:

$$0 \rightarrow \varprojlim_{s_1} F \longrightarrow \varprojlim_{\Lambda_T} F \longrightarrow I_1 / (I_0 \cap I_1) \rightarrow 0.$$

We construct our short exact sequence by letting F_0 be the projective system equal to F , except on $\gamma, \tau_1 \in \Lambda_T$ where we have $F_0(\tau_1) = R$ and $F_0(\gamma) = R/I_0$ and letting K be their kernel system. Our sequence will then look like:

$$0 \rightarrow K \rightarrow \begin{array}{|c|c|} \hline & R/I_0 \\ \hline & R \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline & R/I_0 \\ \hline & R/I_1 \\ \hline \end{array} \rightarrow 0$$

$F_0 \qquad F$

Observe that the vertex in the kernel is $(I_0 + I_1)/I_0 \simeq I_1/(I_0 \cap I_1)$, which gives:

$$0 \rightarrow \begin{array}{ccc} 0 & 0 & 0 \\ | & \diagdown & \diagup \\ 0 & & I_1 \\ & \diagdown & \diagup \\ & I_1/(I_0 \cap I_1) \end{array} \rightarrow \begin{array}{ccc} R & R & R \\ | & \diagdown & \diagup \\ R/I_0 & & R \\ & \diagdown & \diagup \\ & R/I_0 \end{array} \rightarrow \begin{array}{ccc} R & R & R \\ | & \diagdown & \diagup \\ R/I_0 & & R/I_1 \\ & \diagdown & \diagup \\ & R/(I_0 + I_1) \end{array} \rightarrow 0$$

We would like to compute $\varprojlim_{\Lambda^T}^{(1)} K$, and we therefore consider its cellular homology complex:

$$0 \xrightarrow{\delta_2} I_1 \xrightarrow{\delta_1} I_1/(I_0 \cap I_1)$$

We compute the needed homology groups:

$$\begin{aligned} H_1(K) &= \ker(I_1 \rightarrow I_1/(I_0 \cap I_1)) = I_0 \cap I_1 \\ H_0(K) &= \ker(\delta_0)/\text{Im}(\delta_1) = 0/\text{Im}(\delta_1) = 0. \end{aligned}$$

As earlier we have $H_i(K) = \varprojlim_{\Lambda^T}^{(2-i)} K$ from theorem 2.5.2 and from lemma 2.5.7 $\varprojlim_{\Lambda^T} K = 0$. Then from the long exact sequence,

$$\begin{array}{ccccccccccc} 0 & \rightarrow & \varprojlim_{s_1} F & \rightarrow & \varprojlim_{\Lambda^T} F & \rightarrow & \varprojlim_{\Lambda^T}^{(1)} K & \rightarrow & \varprojlim_{s_1}^{(1)} F & \rightarrow & \varprojlim_{\Lambda^T}^{(1)} F & \rightarrow & \varprojlim_{\Lambda^T}^{(2)} K & \rightarrow & 0 \\ & & \parallel & & & & \parallel & & \parallel & & \parallel & & \parallel & & \\ & & R \oplus I_0 & & & & I_1/(I_0 \cap I_1) & & 0 & & & & 0 & & \end{array}$$

it follows that

$$\varprojlim_{\Lambda^T} F \simeq R \oplus I_0 \oplus I_1/(I_0 \cap I_1).$$

□

Chapter 4

The Dimension of Large Complexes

In this chapter we use the theory developed in the earlier chapters to compute the dimension of larger spline spaces, such as the Morgan-Scott triangulation in section 4.2.1 and 4.2.2, which will be the last part of the thesis. We however first need some sort of theory, to split large spline system into smaller pieces. Since we are representing our complexes by posets, we would like this theory to say something about how to split/*degenerate* \varprojlim_{Λ} in a way such that if $\Lambda_1, \Lambda_2 \subset \Lambda$ satisfies $\Lambda_1 \cup \Lambda_2 = \Lambda$, then we could compute \varprojlim_{Λ_i} and then have an expression for \varprojlim_{Λ} in the \varprojlim_{Λ_i} 's. This however, is almost the exact same as one of Laudal's results in [1]. We consider it in the following section.

4.1 Decomposing the inverse limit of a poset

We need a definition, keep in mind the notion of the order complex $\Lambda^{(1)}$ from definition 2.1.17.

Definition 4.1.1. Given two posets Γ and Λ , a κ -**functor of Γ in Λ** is an order-preserving map

$$\kappa : \Gamma \longrightarrow \Lambda^{(1)},$$

such that $\kappa(\gamma) \subseteq \Lambda^{(1)}$ is a closed set for all $\gamma \in \Gamma$.

Let C_{Λ} be the abelian category of projective systems on Λ with values in the category \mathcal{A} of abelian groups. By constructing a map $\Gamma \rightarrow \mathcal{A}$ by

$$\gamma \mapsto \varprojlim_{\kappa(\gamma)} F,$$

we see that a projective system F on Λ induces a projective system on Γ .

Since our goal is to *degenerate* \varprojlim_{Λ} , suppose that

$$\kappa(\Gamma) = \bigcup_{\gamma \in \Gamma} \kappa(\gamma) = \Lambda.$$

Then there are canonical homomorphisms

$$\varprojlim_{\Lambda} F \longrightarrow \varprojlim_{\kappa(\gamma)} F$$

for all $\gamma \in \Gamma$ and this family of homomorphisms defines a canonical homomorphism

$$\varprojlim_{\Lambda} F \longrightarrow \varprojlim_{\Gamma} \varprojlim_{\kappa(\gamma)} .$$

The construction above gives the following lemma.

Lemma 4.1.2. Given a κ -functor $\kappa : \Gamma \rightarrow \Lambda^{(1)}$ satisfying the following two conditions:

1. $\text{Im}(\kappa) = \Lambda$,
2. if $\gamma_1, \gamma_2 \in \Gamma$, and $\lambda \in \kappa(\gamma_1) \cap \kappa(\gamma_2)$, then there exist $\gamma \in \Gamma$ such that $\gamma_1 > \gamma < \gamma_2$ and $\lambda \in \kappa(\gamma)$,

then the spectral sequence

$$E^{p,q} = \varprojlim_{\Gamma} {}^{(p)}\varprojlim_{\kappa(\gamma)} {}^{(q)}$$

converges to

$$\varprojlim_{\Lambda} {}^{\bullet}F .$$

Proof. See proposition 1.3.1, page 270 in [1]. □

4.2 The Morgan-Scott triangulation

In this section we discuss the problem of computing the dimension of the C^r spline space for the planar Morgan-Scott triangulation, Δ_{MS} .

It is known from [13] that the dimension of $S_2^1(\Delta_{MS})$ is either 6 or 7 and in [14] a more general result is shown, namely that $S_n^1(\Delta_{MS}) = \frac{1}{2}n(7n-15)+7$, depending on the geometry of the triangulation. We will reprove this result with use of homological methods, ending up with a clearer geometrical dependence, $0 = \sum_{i=6}^6 \alpha_i L_i^2$.

Let Δ_{MS} denote the Morgan-Scott triangulation of a triangle (figure 4.1), and let $\mathcal{S}_n^r(\Delta)$ be the set of C^{r+1} continuous piecewise polynomial functions of total degree n over Δ .

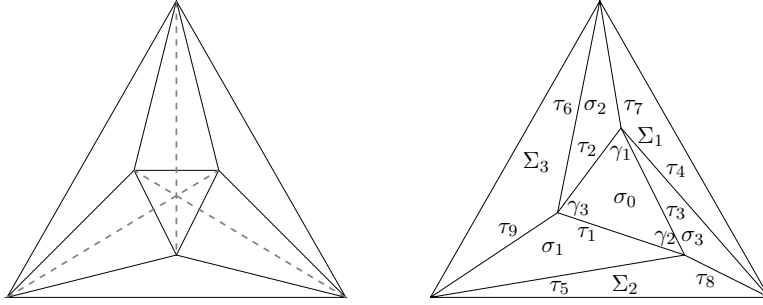


Figure 4.1: Two different Morgan-Scott triangulations Δ_{MS} .

To any triangulation Δ we associate a ranked poset Λ_Δ where cells are nodes and relations are given by incidence. The associated ranked poset of a Morgan-Scott triangulation Δ_{MS} will be denoted Λ_{MS} , and is shown in figure 4.2. In the MS-triangulation there are 7 2-cells, denoted by Σ_i , $i = 1, 2, 3$ or σ_j , $j = 0, 1, 2, 3$, depending on whether it has a boundary facet or not. The inner 1-cells are denoted by τ_k , $k = 1, \dots, 9$ and the inner three vertices by γ_l , $l = 1, 2, 3$.

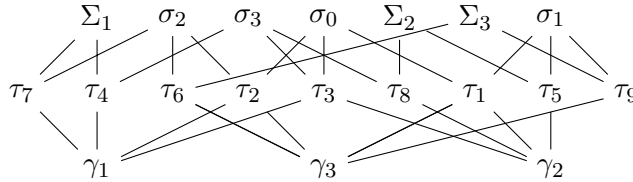


Figure 4.2: The complex of the Morgan-Scott triangulation.

4.2.1 Dimension of the space $S_n^r(\Delta_{MS})$

Computing the dimension of a MS-triangulation isn't straight forward, and has been attempted by several different mathematicians over the years. We saw some of the results in the introduction of this section. Still no one has yet computed $S_n^r(\Delta_{MS})$ for a general r .

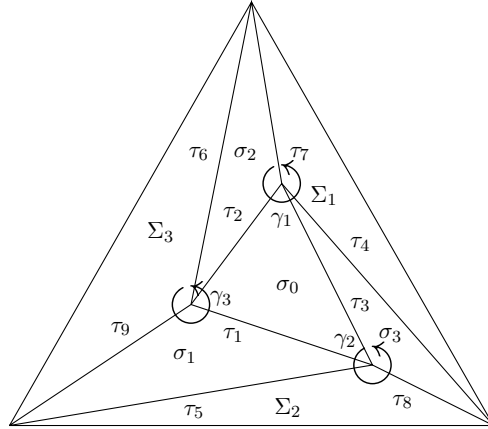
In order to compute the dimension of $S_n^r(\Delta_{MS})$, we are going to introduce a slightly simplified projective system on Λ_{MS} , denoted by G . We shall consider the simplified version as a sub projective system of F with cokernel Q .

The projective system G is obtained from F by annihilating the center triangle, i.e. for all $\lambda \in \Lambda \setminus \sigma_0$ we have $G(\lambda) = F(\lambda)$, but $G(\sigma_0) = 0$. Hence G is not a spline system. It follows that our cokernel Q is given by $Q(\sigma_0) = R$ and 0 elsewhere.

We obviously have $\varprojlim Q = R$, and $\varprojlim^i Q = 0$ for $i > 0$, and

$$0 \rightarrow \varprojlim_{\Lambda_{MS}} G \rightarrow \varprojlim_{\Lambda_{MS}} F \rightarrow R \rightarrow 0.$$

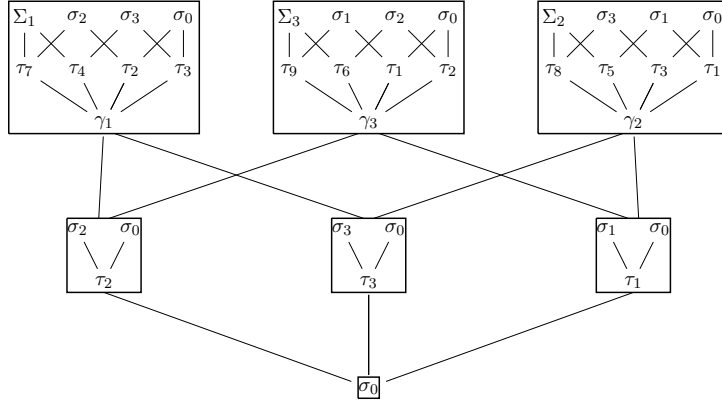
We hence need to compute the inverse limit of G , using the methods developed in section 4.1. Let us take a look at the Λ_{MS} , to see how we can simplify it.



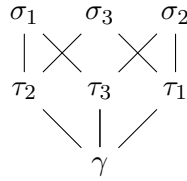
We would like to *decompose* G and are going to do it by splitting it up into three sets, one for each of the \odot 's, by letting each of the sets be $\nabla\gamma_i$. Further we intersect the sets. Hence our *decomposition* will look like:

$$\begin{array}{ccccc} \nabla\gamma_1 & \nabla\gamma_3 & \nabla\gamma_2 & & \\ | & \times & \times & | & \\ \nabla\tau_2 & \nabla\tau_3 & \nabla\tau_1 & & \\ & | & & & \\ & \nabla\sigma_0 & & & \end{array}$$

and expanded out it is the poset:



Let Γ be the poset:



We immediately recognize it as having the same structure as S_2 , the spline star of degree 2.

Now let $\kappa : \Gamma \longrightarrow \Lambda_{MS}^{(1)}$ be given by:

$$\begin{aligned} \kappa(\sigma_i) &= \nabla \gamma_i & \text{for } \sigma_i \in \Gamma \\ \kappa(\tau_i) &= \nabla \tau_i & \text{for } \tau_i \in \Gamma \\ \kappa(\gamma) &= \nabla \sigma_0 & \text{for } \gamma \in \Gamma \end{aligned}$$

From lemma 4.1.2 it follows that we may now compute the dimension of $S_n^r(\Delta_{MS})$ by $\varprojlim_{\Lambda_{MS}} G = \varprojlim_{\Gamma} \varprojlim_{\kappa(\gamma)} G$. We therefore compute:

$\varprojlim_{\nabla \gamma_i} G :$

The projective system G on each of the $\nabla \gamma_i$'s are sub projective systems of F over S_3 , the spline star of degree 3, with cokernel R . Hence:

$$0 \rightarrow \varprojlim_{\nabla \gamma_i} G \longrightarrow \varprojlim_{\nabla \gamma_i} F \longrightarrow R \rightarrow 0.$$

$$\begin{aligned} \varprojlim_{S_3} F &\simeq R \oplus \sum_{i=0}^3 \left(I_{li} \cap \sum_{j=0}^{i-1} I_{lj} \right) \\ &= R \oplus I_{l_1} \cap I_{l_0} \oplus I_{l_2} \cap (I_{l_0} \oplus I_{l_1}) \oplus I_{l_3} \cap (I_{l_0} \oplus I_{l_1} \oplus I_{l_2}). \end{aligned}$$
$$\varprojlim_{\nabla \gamma_i} G \simeq I_{l_1} \cap I_{l_0} \oplus I_{l_2} \cap (I_{l_0} \oplus I_{l_1}) \oplus I_{l_3} \cap (I_{l_0} \oplus I_{l_1} \oplus I_{l_3}),$$

However, it turns out to be hard to calculate the term $I_l \cap (I_i + I_j + I_k)$. So to simplify the calculations we will use an isomorphism correspondence between the ideal sums given by:

$$I_j \cap I_i + I_k \cap (I_i + I_j) + I_l \cap (I_i + I_j + I_k) \simeq (I_i \cap I_j) + [(I_i + I_j) \cap (I_k + I_l)] + (I_k \cap I_l).$$

It follows that:

$$\begin{aligned}\varprojlim_{\nabla\gamma_1} G &= (I_2 \cap I_7) \oplus [(I_2 + I_7) \cap (I_3 + I_4)] \oplus (I_3 \cap I_4) \\ \varprojlim_{\nabla\gamma_2} G &= (I_3 \cap I_8) \oplus [(I_3 + I_8) \cap (I_1 + I_5)] \oplus (I_1 \cap I_5) \\ \varprojlim_{\nabla\gamma_3} G &= (I_1 \cap I_9) \oplus [(I_1 + I_9) \cap (I_2 + I_6)] \oplus (I_2 \cap I_6)\end{aligned}$$

$\varprojlim_{\nabla \tau_i} G :$

From proposition 3.2.3 we have that

$$\varprojlim_{\nabla \tau_i} G = G(\sigma_i) \times_{G(\tau_k)} G(\sigma_0) = R \times_{R/I_k} 0 \simeq I_k.$$

$\varprojlim_{\nabla \sigma_0} G :$

By construction of \varprojlim

$$\varprojlim_{\nabla \sigma_0} G = G(\sigma_0) = 0.$$

We now construct a new poset $\Delta^* = \Gamma \setminus \gamma$:

$$\begin{array}{ccccc} \sigma_1 & & \sigma_3 & & \sigma_2 \\ & \diagdown & & \diagup & \\ & \tau_2 & & \tau_3 & \\ & \diagup & & \diagdown & \\ & \tau_1 & & \tau_2 & \end{array}$$

It then follows from proposition 2.5.6 that we we have

$$\varprojlim_{\Delta^*} L = \varprojlim_{\Gamma} \varprojlim_{\kappa(\gamma)} G \simeq \varprojlim_{\Lambda_{MS}} G,$$

where L is the projective system on Δ^* given by:

$$\begin{aligned} L(\sigma_1) &= (I_2 \cap I_7) + [(I_2 + I_7) \cap (I_3 + I_4)] + (I_3 \cap I_4) \\ L(\sigma_2) &= (I_3 \cap I_8) + [(I_3 + I_8) \cap (I_1 + I_5)] + (I_1 \cap I_5) \\ L(\sigma_3) &= (I_1 \cap I_9) + [(I_1 + I_9) \cap (I_2 + I_6)] + (I_2 \cap I_6) \\ L(\tau_i) &= I_i \end{aligned} \tag{4.1}$$

To find the dimension of $S_n^r(\Delta_{MS})$ we need to compute the cellular homology groups of the projective system L of Δ^* :

$$\begin{aligned} & \left((I_2 \cap I_7) + [(I_2 + I_7) \cap (I_3 + I_4)] + (I_3 \cap I_4) \right) \\ \oplus & \left((I_3 \cap I_8) + [(I_3 + I_8) \cap (I_1 + I_5)] + (I_1 \cap I_5) \right) \\ \oplus & \left((I_1 \cap I_9) + [(I_1 + I_9) \cap (I_2 + I_6)] + (I_2 \cap I_6) \right) \\ & \downarrow \delta \\ & I_2 \oplus I_3 \oplus I_1 \end{aligned} \tag{4.2}$$

To find the last term in $\varprojlim_{\Lambda_{MS}} G$ we need to compute $H^1(L)$, i.e. finding the kernel of the differential δ above. Then the dimension of $S_n^r(\Delta_{MS})$ is given by the relation:

$$\begin{aligned}
\dim S_n^r(\Lambda_{MS}) &= \dim \left(\varprojlim_{\Lambda_{MS}} F \right) \\
&= \dim R + \dim \left(\varprojlim_{\Lambda_{MS}} G \right) \\
&= \dim R + \dim \left(\varprojlim_{\Delta^*} L \right) \\
&= \dim R + \dim (L(\sigma_1) \oplus L(\sigma_2) \oplus L(\sigma_3)) + \dim H^1(L) \\
&\quad - \dim (L(\tau_1) \oplus L(\tau_2) \oplus L(\tau_3)) \\
&= \dim R + 3 \dim L(\sigma_i) - 3 \dim L(\tau_i) + \dim H^1(L)
\end{aligned}$$

We have now actually proven the following theorem:

Theorem 4.2.1. Let $H^1(L) = \ker(\delta)$ in equation (4.2) and $L(\sigma_i), L(\tau_i)$ be as given in equation (4.1). Then:

$$\dim S_n^r(\Delta_{MS}) = 3 \dim L(\sigma_i) - 3 \dim L(\tau_i) + \dim R + \dim H^1(L)$$

To calculate the dimension we need the following result:

Proposition 4.2.2. Given a vector space R and two sub vector spaces U and V such that, neither $U \subseteq V$, nor $V \subseteq U$ and $\dim U + \dim V \geq \dim R$, then the dimension of $U \cap V$ is given as the codimension of the sum of the codimensions of U and V , i.e.

$$\dim(U \cap V) = \dim V + \dim U - \dim R.$$

Proof. See problem 2.12 d), page 13 in [11]. □

Corollary 4.2.3.

$$\dim S_n^r(\Delta_{MS}) = 9 \binom{n+1-r}{2} - 2 \binom{n+2}{2} + \dim H^1(L),$$

where we let $\binom{a}{2} = 0$ if $a \leq 0$.

Proof. From section 2.5 it follows that:

$$\begin{aligned}
\dim R &= \binom{n+2}{2} \\
\dim L(\tau_i) &= \dim(I_i) \\
&= \binom{n+1-r}{2} \\
\dim L(\sigma_i) &= \dim(I_i \cap I_j) + \dim((I_i + I_j) \cap (I_k + I_l)) + \dim(I_k \cap I_l) \\
&= 2 \left[2 \binom{n+1-r}{2} - \binom{n+2}{2} \right] + \left[2 \binom{n+1-r}{2} \right. \\
&\quad \left. - \left(2 \binom{n+1-r}{2} - \binom{n+2}{2} \right) \right] - \binom{n+2}{2} \\
&= 4 \binom{n+1-r}{2} - \binom{n+2}{2}.
\end{aligned}$$

Hence it follows straight from theorem 4.2.1:

$$\begin{aligned}
\dim S_n^r(\Delta_{MS}) &= \dim R + 3 \dim L(\sigma_i) - 3 \dim L(\tau_i) + \dim H^1(L) \\
&= \binom{n+2}{2} + 3 \left[4 \binom{n+1-r}{2} - \binom{n+2}{2} \right] - 3 \binom{n+1-r}{2} \\
&\quad + \dim H^1(L) \\
&= 9 \binom{n+1-r}{2} - 2 \binom{n+2}{2} + \dim H^1(L)
\end{aligned}$$

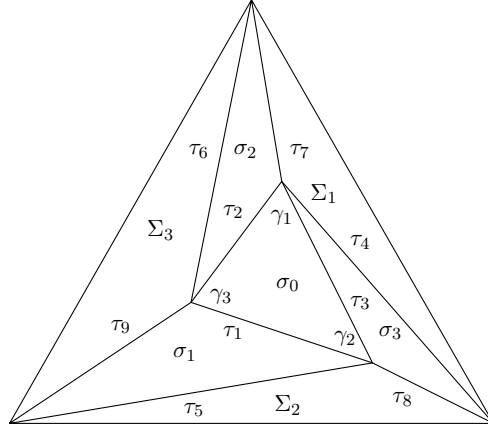
□

Here however, ends this thesis' general result concerning the dimension of the Morgan-Scott triangulation. Computing this kernel has proven to become quite a headache and it is unfortunately not straight forward. It is reasonable to believe that it consists of two terms, one only consisting of ideals depending on n and r and one geometric term depending on the orientation of the inner triangle in the Morgan-Scott triangulation, which we will consider in the next and last section. Although being unable to find a final expression of this kernel, we hope, within reasonable time, to published a final answer to the question of the dimension of the MS-triangulation.

We however give the result proven by several athours before, nameley the dimension of $S_n^1(\Delta_{MS})$.

4.2.2 Dimension of the space $S_n^1(\Delta_{MS})$

The geometric contribution can be found explicitly by observing the cell complex of G , illustrated below:



Proposition 4.2.4. The geometric part of $H_1(L)$ is given by:

$$0 = \sum_{i=1}^6 \alpha_i L_i^{r+1}, \quad (4.3)$$

where $\alpha_i \in k$, and the three restrictions $F(\sigma_1) \in I_1$, $F(\sigma_2) \in I_2$ and $F(\sigma_3) \in I_3$, where L_i is the linear functional corresponding to $F(\tau_i) = (k[x, y]/(L_{\tau_i}^{r+1}))_{\leq n}$.

Proof. Observe from the figure above, that if the elements of Σ_1 and σ_2 should agree on τ_7 , then by lemma 2.5.2

$$\Sigma_1 - \sigma_2 \in \tau_7,$$

which implies

$$G(\Sigma_1) - G(\sigma_2) \in L_7^{r+1}$$

or equivalently

$$-(G(\Sigma_1) - G(\sigma_2)) = \alpha_1 L_7^{r+1},$$

where $\alpha_1 \in k$. The same holds for the other τ_i 's, which gives rise to the following equations:

$$\begin{aligned} -(G(\Sigma_1) - G(\sigma_2)) &= \alpha_1 L_7^{r+1} \\ G(\Sigma_1) - G(\sigma_3) &= \alpha_2 L_4^{r+1} \\ -(G(\Sigma_2) - G(\sigma_3)) &= \alpha_3 L_8^{r+1} \\ G(\Sigma_2) - G(\sigma_1) &= \alpha_4 L_5^{r+1} \\ -(G(\Sigma_3) - G(\sigma_1)) &= \alpha_5 L_9^{r+1} \\ G(\Sigma_3) - G(\sigma_2) &= \alpha_6 L_6^{r+1} \end{aligned} \quad (4.4)$$

Reindexing the L_i 's such that $\{7, 4, 8, 5, 9, 6\} = \{1, 2, 3, 4, 5, 6\}$ and adding the equations in (4.4) together, we obtain:

$$0 = \sum_{i=1}^6 \alpha_i L_i^{r+1}.$$

Further σ_0 and σ_i , for $i = 1, 2, 3$ should agree on τ_i . But since $G(\sigma_0) = 0$, $G(\sigma_i)$ should agree with 0 on I_i . Hence we get the three extra restrictions; $\sigma_1 \in \tau_1$, $\sigma_2 \in \tau_2$ and $\sigma_3 \in \tau_3$. \square

For the case $r = 1$ we can calculate this geometric term explicitly.

Corollary 4.2.5. Assume the field k in $F(\sigma_i) = k[x, y]$ has characteristic $\neq 2$, and that the τ_i 's are in general position. Then the geometric part of $H_1(K)$ from proposition 4.2.4 for $r = 1$ either contributes with 0 or 1 in $\dim H_1(L)$.

Proof. If we write L_i with homogenous coordinates

$$L_i = a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3,$$

then squaring L_i gives

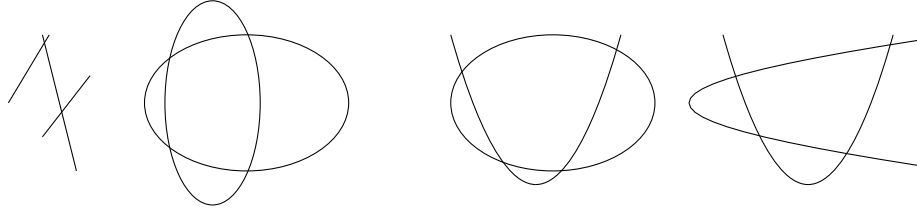
$$\begin{aligned} L_i^2 = & (a_{1i}^2)x_1^2 + (a_{2i}^2)x_2^2 + (a_{3i}^2)x_3^2 \\ & + (2a_{1i}a_{2i})x_1x_2 + (2a_{1i}a_{3i})x_1x_3 + (2a_{2i}a_{3i})x_2x_3. \end{aligned}$$

If $\sum_{i=1}^6 \alpha_i L_i^2 = 0$ either there is a linear relation between the quadratic forms or all of the α_i 's, $i = 1, \dots, 6$, equal zero.

If we look at the six homogenous coordinates as points in the plane, the question reduces to whether or not it is a relation between them, i.e. if they lie on a conic section or not. For 6 **general** points in the plane there doesn't exist a conic section intersecting them all. This means they are linearly independent, and $\sum_{i=1}^6 \alpha_i L_i^2 = 0$ iff. $\alpha_i = 0$ for all $i = 1, \dots, 6$.

However, they may lie on a conic section, but if they do, they are dependent. I.e. if we fix one of the α_i 's we are able to express the rest of them by it, hence there is exactly one relation.

Assume that there exists more than one relation between the points, i.e. they lie on more than one conic section, hence then they lay on at least two. Two conic sections in the plane only share 4 points, hence it is not possible with more than one relation/conic section. To illustrate the impossibility of more than one relation, we include the figure below which shows some intersections of two conic sections.



The leftmost figure would have been a possibility if we could have 5 L_i 's on a line as points in the dual plane, however we assumed that the τ_i 's were in general position.

We conclude that the contribution to $H_1(L)$ is 1 or 0, depending on whether or not the six lines L_i as points in the dual plane lie on a conic section.

□

The last term we need to compute is the term of ideals depending on n and r in $H^1(L)$. We find it by looking at our map δ in equation (4.2). It expresses how all of the cells over/around the γ_i 's are connected with the edges of the inner triangle. When $r = 1$ all terms of higher degree than 1 vanish on the vertices, so we're only left with the three elements k, x, y . There are three vertices, hence the contribution to $\dim H^1(L)$ is 9.

The known result follows:

Theorem 4.2.6. Assume the τ_i 's of the Morgan-Scott triangulation are in general position, then:

$$\dim S_n^1(\Delta_{MS}) = \frac{1}{2}n(7n - 15) + 7$$

with an addition of 1 in $\dim S_n^1(\Delta_{MS})$ if the six lines L_i in $\sum_{i=1}^6 \alpha_i L_i^2$ as points in the dual plane lie on a conic section.

Proof. From corollary 4.2.3 it follows that:

$$\dim S_n^1(\Delta_{MS}) = 9 \binom{n}{2} - 2 \binom{n+2}{2} + \dim H^1(L)$$

As we have seen $\dim H^1(L)$ contributes with 9 in all cases and 1 extra in some geometrical cases depending on the dimension of $\sum_{i=1}^6 \alpha_i L_i^2$. Hence we have

$$\begin{aligned} \dim S_n^1(\Delta_{MS}) &= 9 \frac{n(n-1)}{2} - 2 \frac{(n+2)(n+1)}{2} + 9 \\ &= \frac{1}{2}n(7n - 15) + 7. \end{aligned}$$

□

Appendix A

We will in this appendix illustrate by an example how in some situations cellular homology significantly reduces the amount of calculations needed to compute \varprojlim_{Λ} compared to order cohomology. As our example we will reprove proposition 3.2.7 in another manner.

Let us first make an observation of how we easier find the order cohomology groups of a complex. We let our complex be K^\bullet in the diagram given below, where the differentials δ_K^0 , δ_C^0 and δ_Q^0 are injections. We are interested in computing $H^1(K)$, which is quite hard from the diagram because $H^1(K) = \ker \delta_K^1 / \text{Im } \delta_K^0$.

$$\begin{array}{ccccc}
 0 & & 0 & & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 C^0 & \xrightarrow{\delta_C^0} & C^1 & \xrightarrow{\delta_C^1} & C^2 \\
 \downarrow & & \downarrow & & \downarrow \\
 K^0 & \xrightarrow{\delta_K^0} & K^1 & \xrightarrow{\delta_K^1} & K^2 \\
 \downarrow & & \downarrow & & \downarrow \\
 Q^0 & \xrightarrow{\delta_Q^0} & Q^1 & \xrightarrow{\delta_Q^1} & Q^2 \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0
 \end{array}$$

Since all of the δ^0 's are injections, we can simplify the diagram by dividing C^1 , K^1 and Q^1 , out by $\delta_C^0(C^0)$, $\delta_K^0(K^0)$ and $\delta_Q^0(Q^0)$ respectively. By snake lemma (see proposition 2.10 in [15]) there is a map from the kernel of δ_Q^0 to the cokernel of δ_C^1 , i.e. a map $[H^1(Q) = \ker(\delta_Q^0)] \rightarrow [\text{Coker}(\delta_C^1) = H^2(C)]$. As shown in the figure on the next page.

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(C) & \longrightarrow & C^1/\delta_C^0(C^0) & \xrightarrow{\delta_C^1} & C^2 & \longrightarrow & H^2(C) \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(K) & \longrightarrow & K^1/\delta_K^0(K^0) & \xrightarrow{\delta_K^1} & K^2 & \longrightarrow & H^2(K) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(Q) & \longrightarrow & Q^1/\delta_Q^0(Q^0) & \xrightarrow{\delta_Q^1} & Q^2 & \longrightarrow & H^2(Q) \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

(A curved arrow connects the $H^1(Q)$ node to the $H^2(C)$ node.)

We are now ready to reprove proposition 3.2.7 using order cohomology. We first render the proposition:

Proposition .0.7. Given the spline star, S_1 , of degree 1 and our usual spline system F , then:

$$\varprojlim_{S_1} F \simeq R \oplus (I_0 \cap I_1)$$

Proof. We start by making our short exact sequence:

$$\begin{array}{ccccccc}
0 & \rightarrow & \begin{array}{cc} 0 & 0 \\ | & | \\ I_0 & I_1 \\ \diagdown & / \\ (I_0 + I_1) \end{array} & \rightarrow & \begin{array}{cc} R & R \\ | & | \\ R & R \\ \diagdown & / \\ R \end{array} & \rightarrow & \begin{array}{cc} R & R \\ | & | \\ R/I_0 & R/I_1 \\ \diagdown & / \\ R/(I_0 + I_1) \end{array} \rightarrow 0
\end{array}$$

corresponding to

$$0 \longrightarrow K \longrightarrow F_0 \longrightarrow F \longrightarrow 0$$

We examine the order cohomology complex of the kernel K .

$$I_0 \times I_1 \times (I_0 + I_1) \rightarrow 2I_0 \times 2I_1 \times 4(I_0 + I_1) \rightarrow 4(I_0 + I_1)$$

We construct a diagram by projecting down to the closed subset $\Lambda_Q = S_1 \setminus \sigma_2$ of S_1 :

$$\begin{array}{ccccc}
0 & \xleftarrow{\delta_C^0} & I_0 \times I_1 \times (I_0 + I_1) & \xrightarrow{\delta_C^1} & 2(I_0 + I_1) \\
\downarrow & & \downarrow & & \downarrow \\
I_0 \times I_1 \times (I_0 + I_1) & \xleftarrow{\delta_K^0} & 2I_0 \times 2I_1 \times 4(I_0 + I_1) & \xrightarrow{\delta_K^1} & 4(I_0 + I_1) \\
\downarrow & & \downarrow & & \downarrow \\
I_0 \times I_1 \times (I_0 + I_1) & \xleftarrow{\delta_Q^0} & I_0 \times I_1 \times 3(I_0 + I_1) & \xrightarrow{\delta_Q^1} & 2(I_0 + I_1) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

It is commutative by lemma 3.1.3.

We now do the same for this diagram as we did for the one above; construct a new diagram by dividing the second column out by the map of the first column.

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(C) & \longrightarrow & I_0 \times I_1 \times (I_0 + I_1) & \xrightarrow{\delta_C^1} & 2(I_0 + I_1) & \longrightarrow & H^2(C) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(K) & \longrightarrow & 2I_0 \times 2I_1 \times 4(I_0 + I_1) / \text{Im } \delta_K^0 & \xrightarrow{\delta_K^1} & 4(I_0 + I_1) & \longrightarrow & H^2(K) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(Q) & \longrightarrow & 2(I_0 + I_1) & \xrightarrow{\delta_Q^1} & 2(I_0 + I_1) & \longrightarrow & H^2(Q) \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

First we note that $H^1(Q) = 0 \Rightarrow H^2(K) \simeq H^2(C)$, and secondly we recognize $H^1(C)$ as $\ker \delta_C^1$. But

$$\begin{array}{ccc}
\ker [I_0 \times I_1 \times (I_0 + I_1) \rightarrow 2(I_0 + I_1)] & \simeq & \ker [I_0 \oplus I_1 \rightarrow (I_0 + I_1)] \\
\parallel & & \parallel \\
H^1(K) & & H_1(K)
\end{array}$$

Hence we have done all this work just to end up with having to compute $H_1(K)$. There is no doubt that cellular homology has a great advantage to order cohomology in this case; by using cellular homology we would have been able to write out $I_0 \oplus I_1 \rightarrow (I_0 + I_1)$ directly from the the kernel system K .

The result will follow as in the proof of proposition 3.2.7, from the long exact sequence of the $\varprojlim_{S_1}^{(j)}$'s. We skip it here, since our goal was to illustrate the difference in amount of work to be done. \square

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